Real Analysis - I

Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 10.3
The Cauchy Criterion

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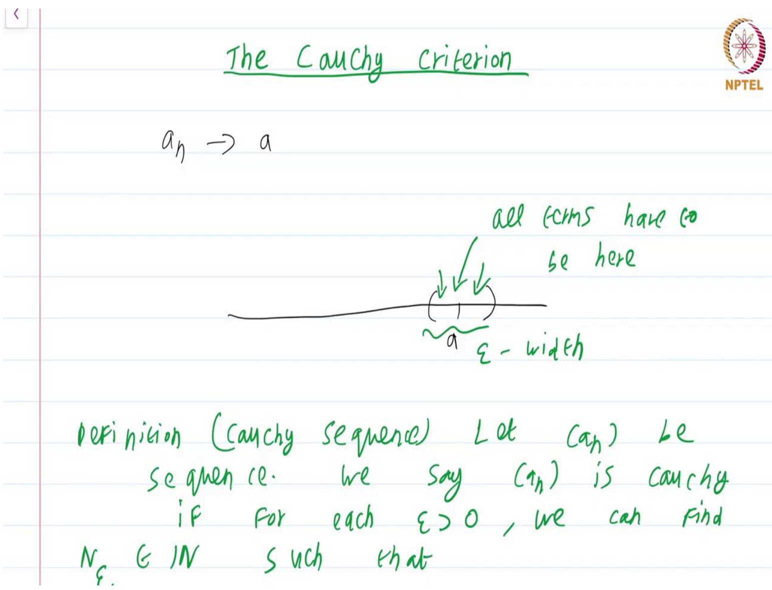
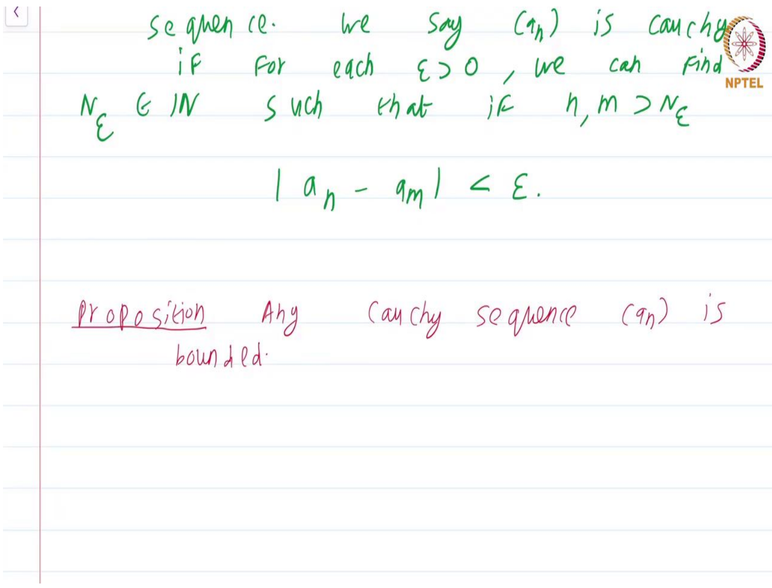


Imagine you have a convergent sequence a_n converging to the point a. Now, observe that what this means is that the sequence gets closer and closer to a. So, if a were to be here on the number line, then eventually all the terms (a_n) will be in any pre-specified interval around the point a. So, after a point, all the terms of the sequence have to be here.

Now, how far along the sequence you have to go to make sure that all the terms in the sequence are indeed in this pre-specified interval depends on the particular sequence. But nevertheless it is always the case that whenever you have a convergent sequence, all the terms have to cluster around this point a in any pre-specified interval.

What this means is that since the points are clustering near a, they have to get closer to each other as well because if you specify this preset interval to be of ϵ width. That means, after a particular point in the sequence all the terms are within 2ϵ width of each other.

If all the points are here, you can see clearly that the terms have to be at the max 2ϵ close to each other. This is what is known as the Cauchy property of a sequence. We will define it precisely. Now, this is the definition of a Cauchy sequence. (Refer Slide Time: 02:40)



Definition: Let (a_n) be a sequence. We say a_n is Cauchy if for each $\epsilon > 0$, we can find $N_\epsilon \in \mathbb{N}$ such that if $n > N_\epsilon$, then so far it looks like the definition is exactly the same.

So, what we want to say now is not just that if $n > N_\epsilon$, what we want to say is we want to sample two different points in the sequence beyond N_ϵ. So, we choose $n, m > N_\epsilon$, then $|a_n - a_m| < \epsilon$.

So, what this is saying is precisely what is illustrated in this picture if you can choose N_ϵ in such a manner that if you would sample two points, any two points of the sequence that lie beyond N_ϵ, then they will be ϵ close to each other at max, then such a sequence is said to be a Cauchy sequence.

Now, we immediately prove a very simple proposition. This should be reminiscent of a similar property that we had established for convergent sequences.

Any Cauchy sequence (a_n) is bounded.

Now, the proof should also remind you of the proof that convergent sequences are bounded. The proof is almost exactly word for word the same.

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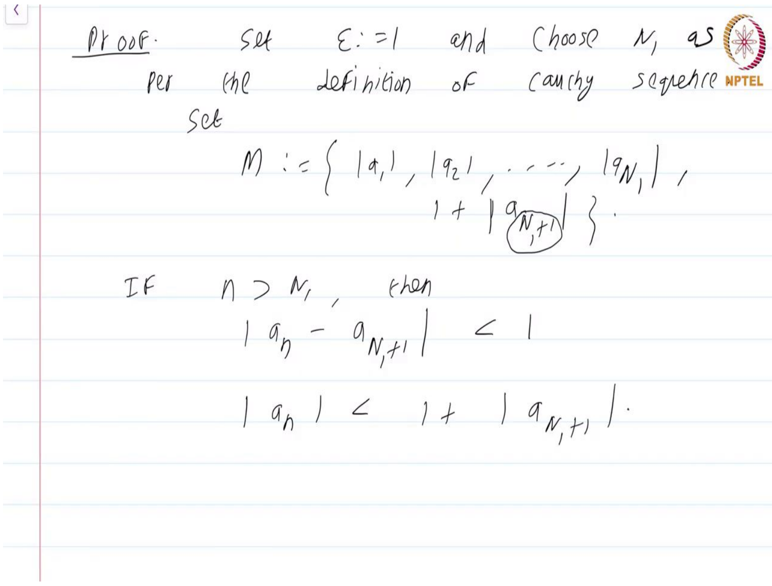

Proof: Set $\epsilon := 1$ and choose N_1, as per the definition of a Cauchy sequence. Now, I have to tell you a candidate bound for the sequence to be $|a_1|, |a_2|, ., |a_{N_1}|, 1 + |a_{N_1+1}|$.

So, I go all the way up till $|a_{N_1}|$, then I add another term to this collection that is $1 + |a_{N_1+1}|$. This is a subscript $N_1 + 1$. So this $N_1 + 1$ is there as a subscript there is a modulus here .

Why does this work? This seems like a weird choice. Well, it follows immediately from the way we have chosen N_1 if $n > N_1$, then we know that $|a_n - a_{N_1+1}| < 1$. Why is that the case because we had chosen ϵ to be 1 and chosen N_1 as per the definition of the Cauchy sequences.

That means, whenever n is small, $n > N_1$, I mean $n, m > N_1$, the difference between a_n and a_m in absolute value has to be less than ϵ. That is the way N_1 was chosen.

So, we must have $|a_n - a_{N_1+1}| < 1$ because both $N_1 + 1$ and n are greater than N_1. Now, in other words, $|a_n| < 1 + |a_{N_1+1}$.

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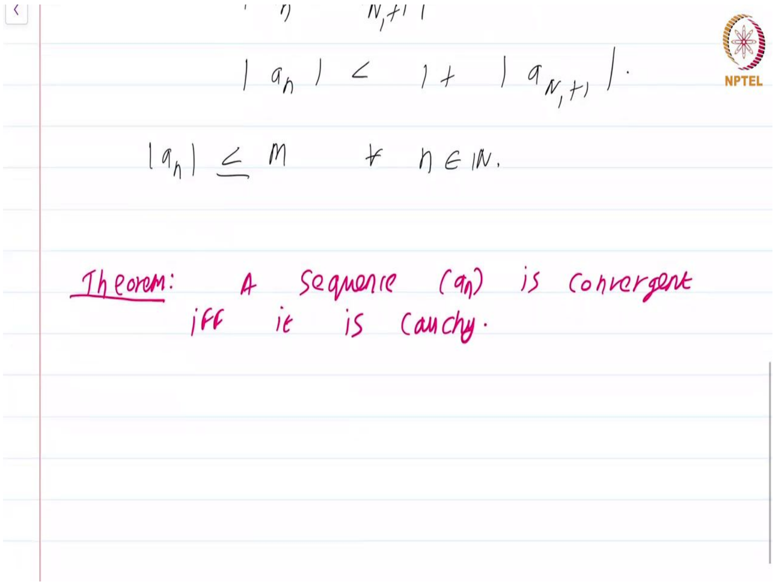

This immediately shows that $|a_n| < M \ \forall n \in \mathbb{N}$. By the very choice of M, this has to be true.

So, this proof is almost word for word exactly the same as what we saw for convergent sequences. Now, the definition of Cauchy sequence came from our intuition that from the fact that convergent sequences, the terms of the sequence must cluster towards the converging point a therefore, must cluster towards each other.

Let us prove that formally, but there is an interesting twist theorem.

A sequence (a_n) is convergent if and only if it is Cauchy.

One direction is palatable. That is how Cauchy sequences were in fact defined it would be a travesty if it turns out that convergent sequences are not Cauchy. But the converse is also true. Cauchy sequences are convergent or the first part as you can expect one direction will be easy.

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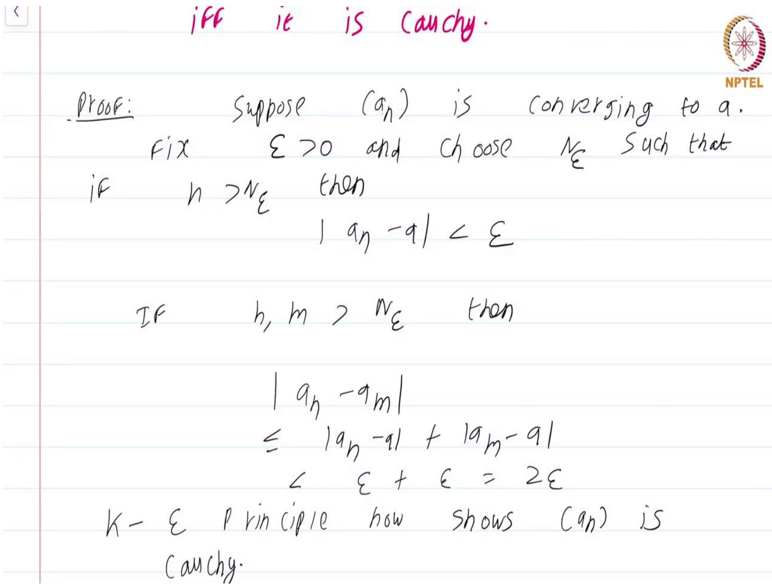

So, suppose a_n is convergent. Our goal is to show that it is Cauchy . Fix $\epsilon > 0$ and choose N_ϵ such that if $n > N_\epsilon$, then $|a_n - a| < \epsilon$.

So, suppose a_n converging to a. Let me just change that to extract this precision converging to a . So, all we have done is we have chosen the element N_ϵ such that if $n > N_\epsilon$, $|a_n - a| < \epsilon$.

Now, what we do is why waste our efforts in thinking, let us just see what will happen if I choose n, m both to be greater than N_ϵ and then, consider their absolute difference $|a_n - a_m|$. Now, I need to compare a_n and a_m, it is not really clear how I do that, but there is only one way to do it that is to introduce the element a. So, this is less than or equal to $|a_n - a| + |a_m - a|$.

I have just added and subtracted the limit a, but both n and m are greater than N_ϵ and N_ϵ was chosen. So, from the definition of the convergence of $a_n \to a$. So, this will have to be $< \epsilon + \epsilon = 2\epsilon$ and this combined with the $K - \epsilon$ principle now shows (a_n) is Cauchy immediately .

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On the other hand, suppose (a_n) is Cauchy. We need a candidate limit.

As (a_n) is bounded, the Bolzano-Weierstrass theorem says some subsequence, say $a_{n_k} \longrightarrow a$.

Claim: $a_n \to a$.

Fix $\varepsilon > 0$, we can find N_ε such that if $n, m > N_\varepsilon$ then

$$|a_n - a_m| < \varepsilon.$$

Now, the converse is a bit involved because we want to show that the sequence that is Cauchy is automatically convergent and the definition of convergence involves this point limit a and given an arbitrary sequence, it is not clear how to determine the limit. There is no algorithm for finding out the limit in any easy manner.

So, let us start the proof. On the other hand, suppose a_n is Cauchy. Take a Cauchy sequence, we need a candidate for the limit. But how do we get this candidate? Well, think about it for a moment. If the sequence (a_n) were to be convergent, then we already know that any subsequence of (a_n) must converge to the same limit.

So, what we will do is we will invoke the Bolzano Weierstrass theorem to say that there must be at least one convergent subsequence whatever the limit of that sequence is we will take that as the candidate limit and then reverse engineer and try to show that the whole sequence (a_n) itself converges to the same limit.

So, we need a candidate limit. So, what we will do is as (a_n) is bounded. That is what we just established in the previous proposition. The Bolzano Weierstrass theorem says some subsequence say a_{n_k} converges to some limit a .

So, our goal is now to show that claim $(a_n) \to a$. Now, how we are going to show this claim. Well that is not really hard. Recall that a subsequence is a sampling of the sequence in the same order right. So, what this has shown is that after a particular point, there are at least some elements which get really close to a, but Cauchy sequence says that if you go along the sequence all the terms cluster along each other.

So, what do we use this for? We use this subsequence as an anchor to make all the terms cluster around terms of the subsequence, but since the subsequence converges to a, all the terms of the sequence must also cluster around a. This is basically a rough and loose idea as to what I am about to do now .

So, fix $\epsilon > 0$, we have to find a candidate N_ϵ. What we do is we can find N_ϵ such that if $n, m > N_\epsilon$, then $|a_n - a_m| < \epsilon$, this is the very definition of a Cauchy sequence.

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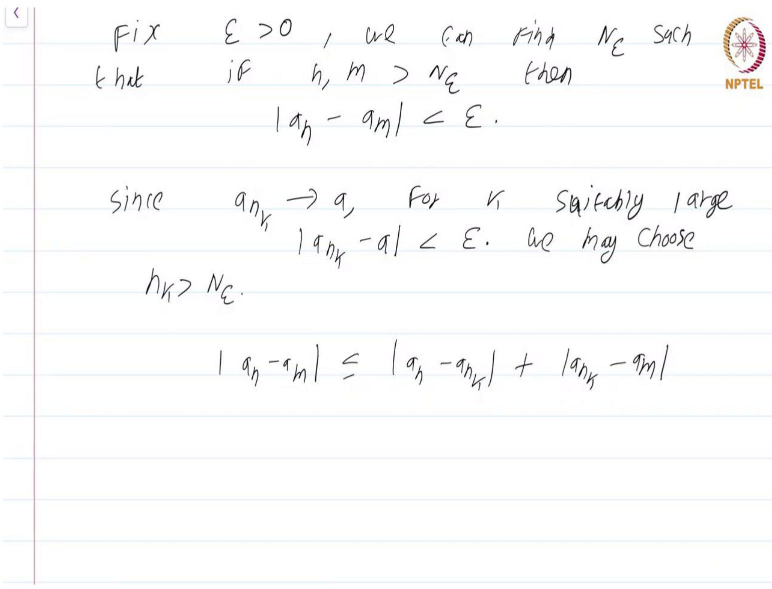

Now, since $a_{n_k} \to a$, for k suitably large $|a_{n_k} - a| < \epsilon$. Now, we may choose $n_k > N_\epsilon$. Remember since $a_{n_k} \to a$, all the terms in the sequence must be close to a for k suitably large. In particular if you choose k to be extremely large, we can make it great we can make $n_k > N_\epsilon$ as well.

Now, $|a_n - a_m|$. So, this is now the central trick of the day is less than or equal to $|a_n - a_{n_k}| + |a_{n_k} - a_m|$. I am using this n_k to anchor the sequence, terms of the sequence . Now, because n_k has been chosen so that $|a_{n_k} - a| < \epsilon$. Oh just one moment I made a in word inadvertent error .

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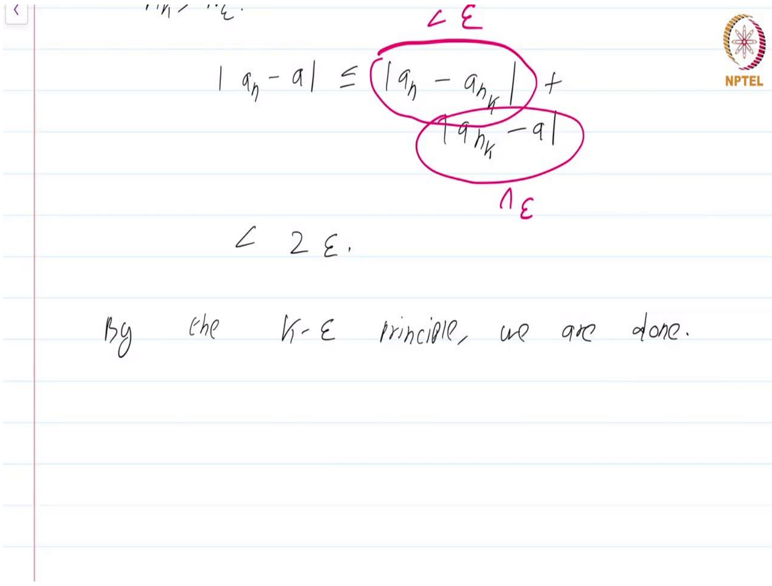

What I want to say is $|a_n - a|$. See I have already dealt with $|a_n - a_m|$ it is less than ϵ. I want to show that $(a_n) \to a$. Sorry about this error. $|a_n - a| \leq a_n - a_{n_k}| + |a_{n_k} - a|$. I have used (a_{n_k}) now to anchor the sequence (a_n).

Now, the first term is certainly going to be less than ϵ simply because that is the way n_k has been chosen. The second term will also be less than ϵ because that is the way n_k has been chosen again right. So, this whole thing is less than 2ϵ. So, by the $K - \epsilon$ principle we are done. So, this shows that a Cauchy sequence is convergent.

In the chapter on topology, we will see that Cauchy sequences play a very very big role. This is a course on Real Analysis and you have just watched the module on the Cauchy Criterion.