Real Analysis - I
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Lecture – 10.2
Bolzano-Weierstrass Theorem

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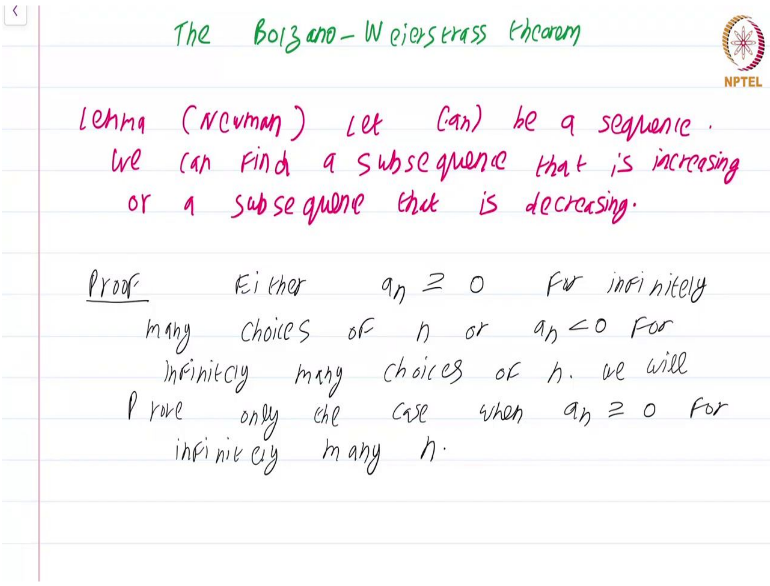

This module is on the famous Bolzano Weierstrass theorem, which is one of the most important theorems in all of analysis. So, please grab a cup of coffee and be up to your optimum concentration levels. We are going to give two different proofs of this Bolzano Weierstrass theorem.

This theorem states that any bounded sequence must have a convergent subsequence. Convergence sequences are bounded, we have already seen that, but the converse is not true, it's only partially true. We are going to provide two different proofs of this result. The first proof relies on an ingenious observation by Donald Newman in 1953.

This theorem is around 100 years older than that, but nevertheless this observation was made only very recently relatively speaking. So, that states the following lemma, this is due to Newman. This states that

Let a_n be a sequence, we can find a subsequence that is increasing or a subsequence that is decreasing.

So, any sequence no matter how complicated and bizarre it is will either have an increasing subsequence or a decreasing subsequence or most probably both . Now, first we will make a simplification. It is going to be proof by cases, where I am going to leave one of the cases to you .

Either $a_n \geq 0$, for infinitely many choices of n or $a_n < 0$, for infinitely many choices of n. In all likelihood both might happen as well . It is possible that a sequence has infinitely many positive terms and infinitely many negative terms , but the catches at least one of these properties must hold.

We will prove only the case when $a_n \geq 0$, for infinitely many choices of n. The other cases are exactly similar and I am going to leave that to you.

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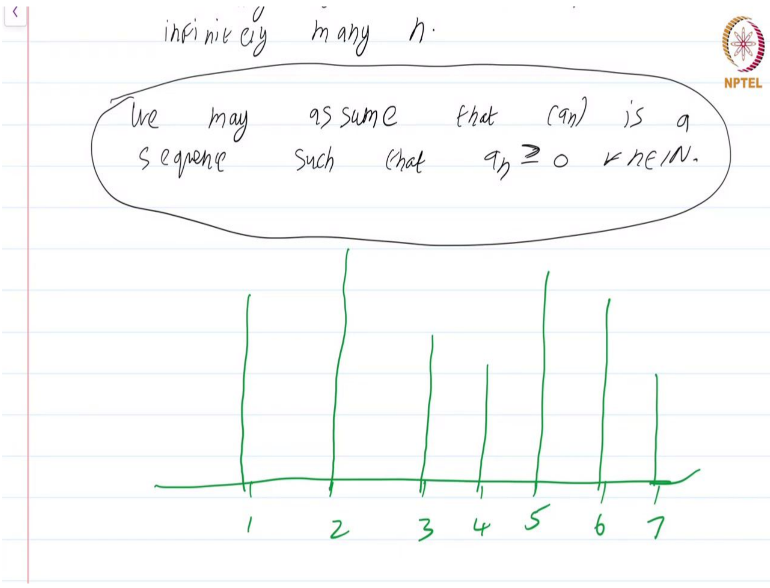

In other words we can make the assumption, we may assume that a_n is a sequence such that $a_n \geq 0, \ \forall n \in \mathbb{N}$. What have I done here? Well, again remember what we are trying to prove.

We are given some arbitrary sequence a_n, we do not know its behaviour, it could be the case that infinitely many terms are positive or infinitely many terms are negative, something one of these definitely has to happen.

I am going to prove it only for those sequences that have $a_n \geq 0, \ \forall n \in \mathbb{N}$. Why am I done if I do it only for these sequences? Well, look at the original sequence a and b started out with. We have first made the assumption that $a_n \geq 0$ for infinitely many n.

In other words we can find a subsequence of a_n such that all terms in that subsequence are greater than or equal to 0. What I have essentially done is; I have removed the subsequence from the original sequence a_n and created a new sequence and by abuse of notation call this new subsequence also as a_n and I am going to prove the result only for this (a_n).

So, essentially what I have done is I have extracted a subsequence, reindex that subsequence and I am going to consider only that subsequence. So, please contemplate for a few minutes after pausing the video. Why if you show only for this case, we are done. Why can we make this assumption ok? It is not hard, it is just basic logic.

Now, we have made the assumption that a_n is a sequence such that $a_n > 0, \ \forall n$. We are going to show that this sequence must either have a subsequence that is increasing or a subsequence that is decreasing.

Now, what we are going to do is we are going to imagine that the sequence is plotted as a graph . So, imagine we have 1, 2, 3, 4, 5 and so on and imagine you are plotting the sequence. So, this might be a_1. I am plotting it as a stick of course, this might be a_2, this might be a_3, this might be a_4, this might be a_5, then so on and so forth. Let me draw a few more terms for representation .

Now, imagine that you have these sticks laid out in front of you and you climb to the top of one stick and you are standing here on the stick.

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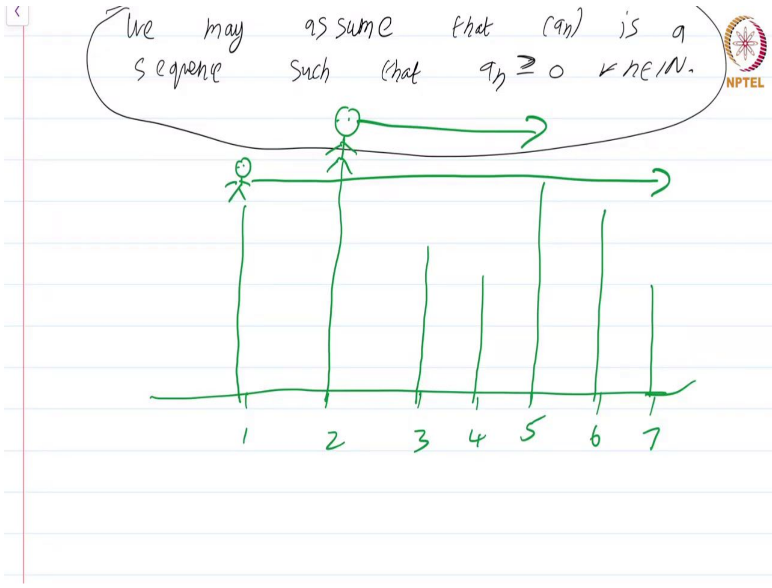

Don't ask me how you are able to stand on an infinitely small stick; this is just for illustrative purposes. Now, imagine that you are standing on top of one of the sticks and you look to your right.

What can happen? Well, there are two possibilities. Either every stick to your right is smaller or at the max as large as the stick you are standing on or this is in the case, right. So, assume for the time being that you are standing on a stick such that every stick that comes later is no larger than the stick you are standing on.

For instance, if you are standing on a_2, this is the case. Every stick that comes later is at the max the size of the stick you are standing on. When you are looking your right from a_2, but if you are in a_4, this is not the case. If you are standing on top of a_4 your view is blocked by a_5, a_6 and by the way not a_7, by the way I have drawn it a_7 is slightly smaller.

So, if you are standing on top of a stick and looking to your right either there will be no stick that is larger than the stick you are standing on or this is not the case. Now, is the key fact.

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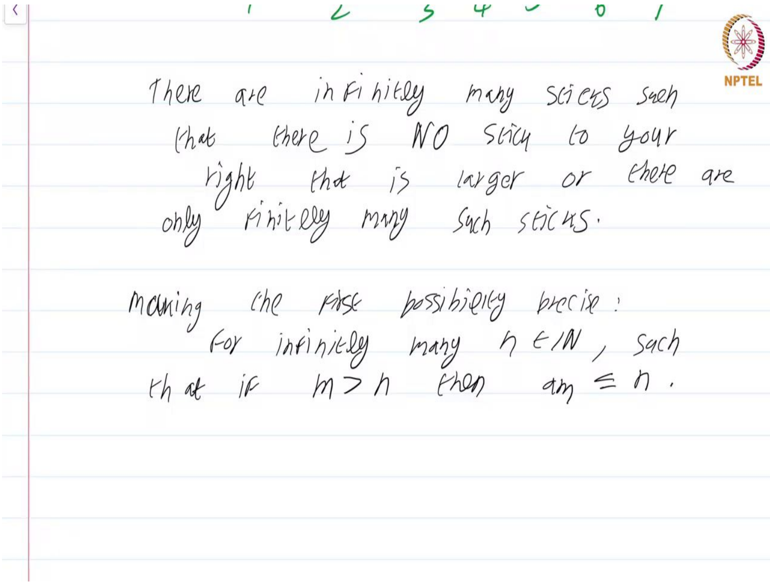

There are infinitely many sticks, such that there is no stick to your right that is larger or there are only finitely many such sticks. There are only two possibilities; either that the collection of sticks, which has the property that when you look to your right, no other stick is larger. This collection is finite or it is infinite .

So, please this is the heart of the proof and this is that ingenious trick that I was talking about. Please pause the video and contemplate this for a few minutes to make sure that you understand that these are the only two possibilities entirely.

Now, we will make the first possibility precise. Making the first possibility precise makes the first possibility precise for infinitely many n in $\mathbb{N}$, such that if $m > n$ then $a_m \leq a_n$.

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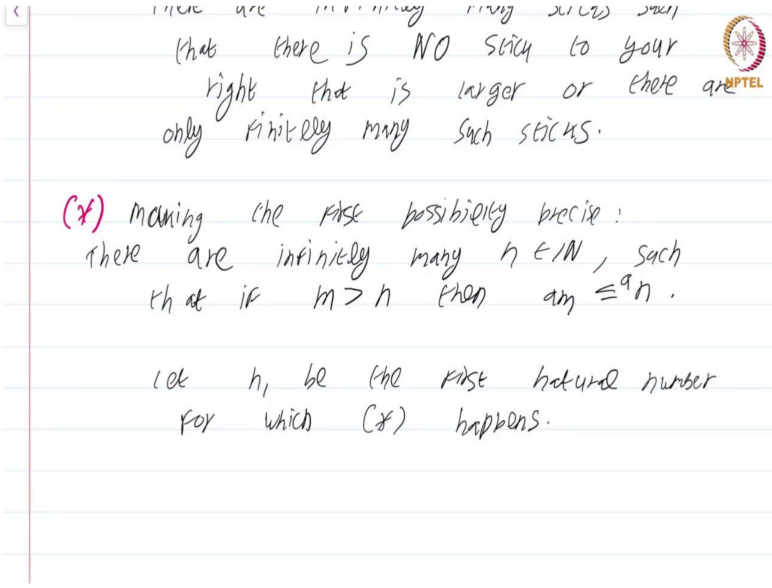

There are infinitely many n, such that if m is greater than there are; let me reword this to be grammatically precise. There are infinitely many natural numbers such that if $m > n$ then $a_m \leq a_n$; this is the first possibility.

Now, I am going to construct for you a decreasing subsequence of a_n using this possibility. How do I do that? Let n_1 be the first natural number for which let me just label the *, for which * happens.

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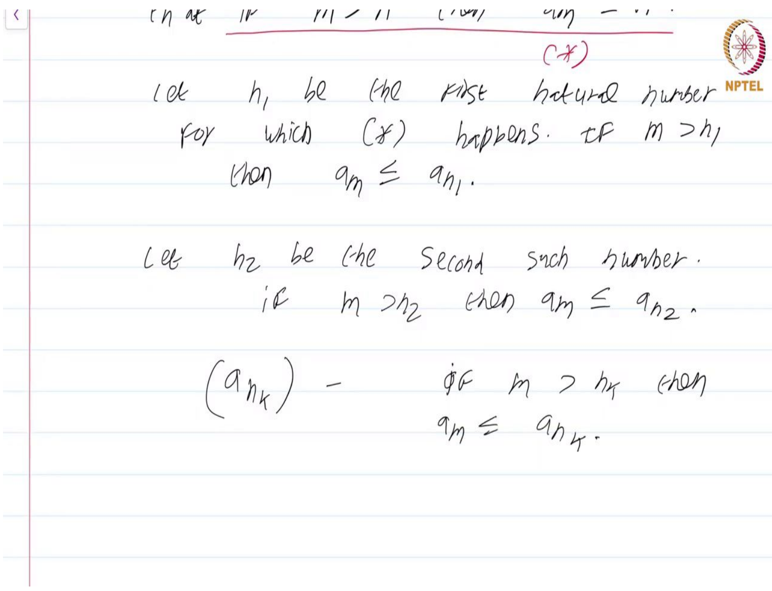

Let us label this as what I am going to call *, not the whole statement. Let n_1 be the first natural number for which * happens. That means, if $m > n_1$ then $a_m \leq a_{n_1}$. So, we have found the first term in the subsequence.

Let n_2 be the second such number either second such number that means, if $m > n_2$ then $a_m \leq a_{n_2}$. We have assumed that there are infinitely many such numbers, so, we can keep iterating this procedure and get a subsequence a_{n_k} .

What is the special property of the subsequence? Well, if $m > n_k$ then $a_m \leq a_{n_k}$. This is the specific feature of the subsequence.

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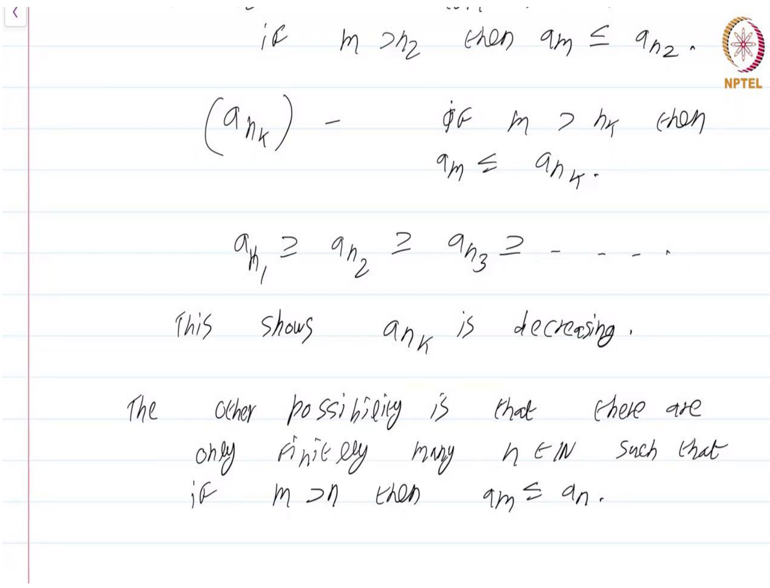

In other words, by construction $a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq \cdots$. That is, how the sequence has been constructed. This shows a_{n_k} is decreasing. So, in this scenario that there are infinitely many sticks such that when you look to your right no stick is larger than the stick you are standing on. We have shown that there is a decreasing subsequence.

The other possibility is that there are only finitely many such sticks, only finitely many n in $\mathbb{N}$, such that if $m > n$ then $a_m \leq a_n$. That means, after a point when you climb on top of a stick and look to your right there will always be at least one stick that is larger than the stick you are standing on.

No matter where you stand, after a particular point; there will be some finitely many sticks for which this is not true. Let us say till the stick 6^{th} stick, it you will be able to you will not be able to find the larger stick to your right, but after a point every stick that comes will have the property that when you look to your right there will be some stick at least that is larger. And I am hoping that already you realize the proof, I am going to leave it as an exercise because it is easy.

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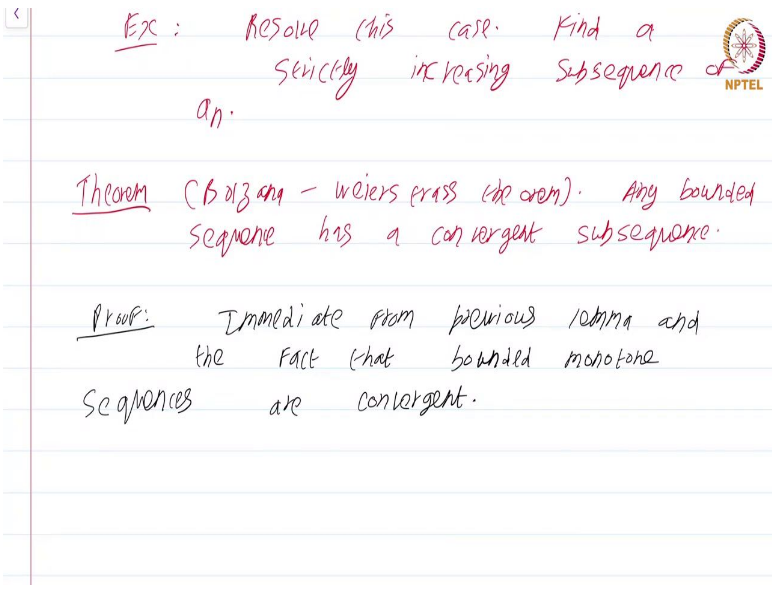

Resolve this case. That is, find a strictly increasing subsequence of a_n. Now, you will be able to prove this quite easily. Once you are done with this proof, please sit down and think .

When there are infinitely many negative terms we have assumed that there is a subsequence that consists of non negative terms. Just think why even if that is not true that you have infinitely many negative terms, you can construct a similar argument.

So, this is an interesting lemma. Let us prove the Bolzano Weierstrass theorem using this lemma.

Theorem: (Bolzano Weierstrass theorem) Any bounded sequence has a convergent subsequence.

Proof: Immediate from the previous lemma and the fact that bounded monotone sequences are convergent. That was something we proved in the last module.

That any increasing bounded sequence is convergent, I left the decreasing case for you. You start with the sequence from the previous lemma; it either has an increasing subsequence or a decreasing subsequence. Apply the fact that any monotone sequence that is bounded is convergent to this particular subsequence, which is increasing or decreasing and we are done.

Now, because this result is so important and because the first proof is a bit tricky we shall provide a more classical proof which is a bit long, but nevertheless worth learning .

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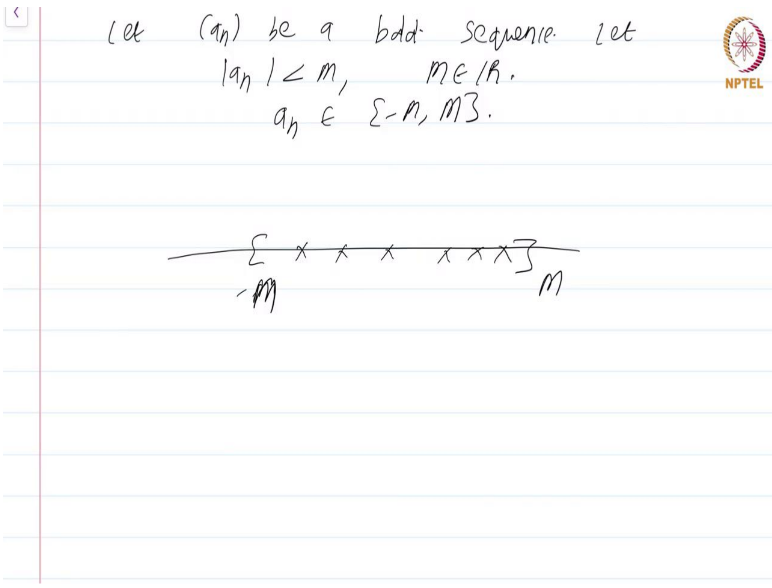

Now, let a_n be a bounded sequence. This is a second proof. Let $|a_n| < m,\ m \in \mathbb{R}$. We have fix the real number which is a bound. In other words a_n is all in the set [-m, +m]. That is what $|a_n| < m$. So, you have [-m,m] and all the terms of the sequence are in here .

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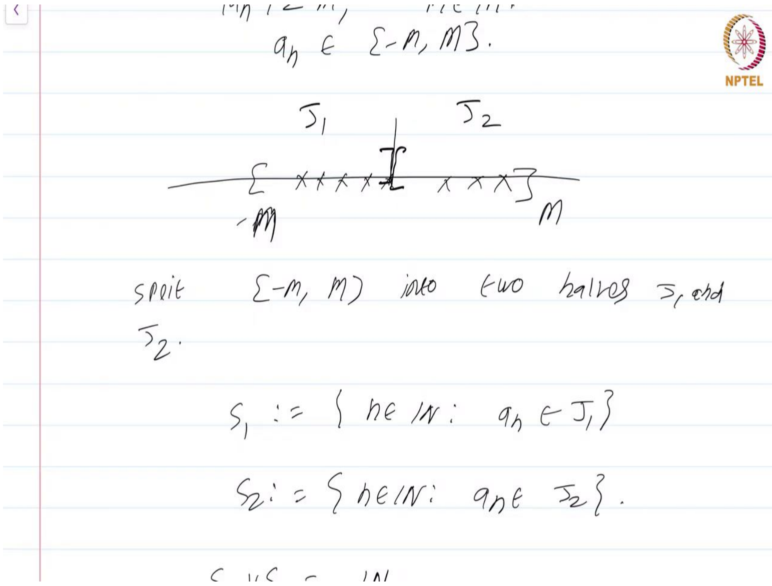

Now, split this into two intervals right in the middle. Call this one J_1, call this one J_2. So, spilt [-m,m] into two halves J_1 and J_2 closed intervals. Both J_1 and J_2 are closed intervals so it is like this. I am just splitting right in the middle.

So, here actually the midpoint will be 0 because it is symmetric about the origin and consider these two sets. $S_1 := \{n \in \mathbb{N} : a_n \in J_1\}$ and $S_2 := \{n \in \mathbb{N} : a_n \in J_2\}$. The key observation is that $S_1 \cup S_2 = \mathbb{N}$.

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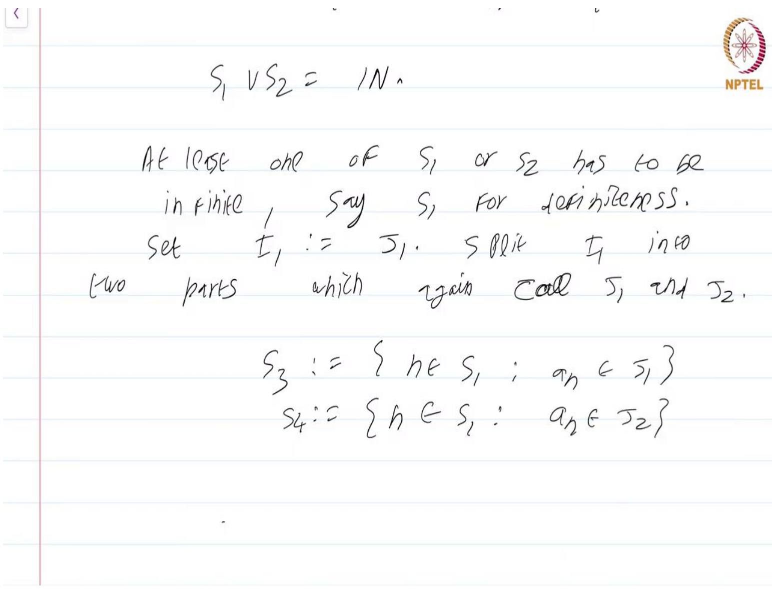

Because $S1 \cup S_2 = \mathbb{N}$,at least one of S_1 or S_2 has to be infinite. Say S_1 for definiteness. It could be either I am just taking it to be S_1 for setting for being definite. Now, set $I_1 := J_1$. So, what have I done?

There are terms of the sequence a_n that fall either in J_1 or J_2. I am looking at those indices for which a_n falls in J_1 and those indices for which a_n falls in J_2, for some either for S_1 or S_2 in a_n should be in J_1 or J_2 infinitely of n. Note, you must not make the mistake of thinking that there are infinitely many points in J_1 such that a_n is. I mean infinitely many points a_n are there in J_1 that is not true.

It could be the case that a_n's repeat. What is crucial is that the collection of indices for which $a_n \in J_1$ is infinite , not the terms of the sequence. It could happen that the terms of the sequence are all the same. There is only one point in J_1 that can also happen, it is just that there must be infinitely many indices.

Now, set $I_1 := J_1$ and now split I_1 into two parts which again I call J_1 and J_2 by abuse of notation, call J_1 and J_2. I am splitting I_1 again into two parts. And just to not overburden the notation I am calling it J_1 and J_2 again. Now, define these two sets $S_3 := \{n \in S_1 : a_n \in J_1\}$ and $S_4 := \{n \in S_1 : a_n \in J_2\}$.

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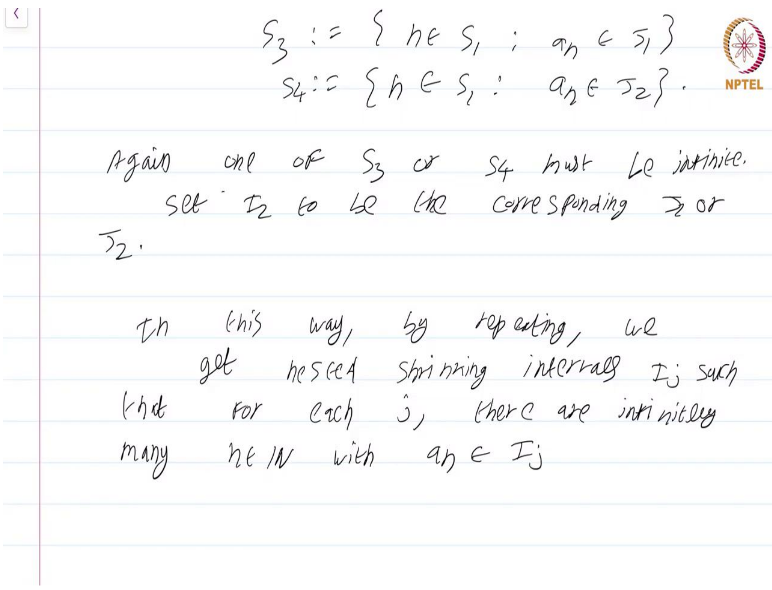

Now, S_1 was an infinite set and $S_3 \cup S_4 = S_1$. Again one of S_3 or S_4 must be infinite. It could be the case that both are also. And set I_2 to be the corresponding J_1 or J_2 . If S_3 is infinite, set $I_2 := J_1$, if S_4 is infinite set $I_2 := J_2$.

Now, it must be clear what needs to be done. In this way repeating this, in this way by repeating that is split into two intervals, look at those indices for which a_n is there in the first part and look at those indices for which a_n is there in the second part, one of these sets must be infinite. Either I mean S_5 and S_6; one of them must be infinite so on. When you repeat this argument you will always get intervals each of which has infinitely many indices, such that a_n belongs to that interval.

So, in this way by repeating we get nested shrinking intervals I_j, such that for each j, there are infinitely many natural numbers with $a_n \in I_j$. I am just continuously splitting and choosing that subinterval which has infinitely many indices such that $a_n \in I_j$.

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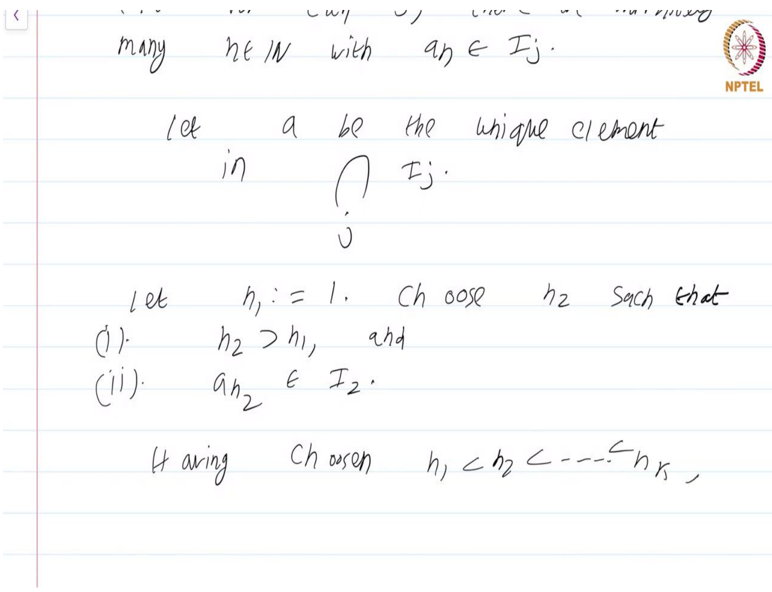

Let a be the unique element in $\cap_j I_j$. There will be an unique element because this is nested shrinking intervals. At each stage the length of the interval is being halved. Now, let $n_1 := 1$. Choose n_2, such that first of all $n2 > n_1$ and second $a_{n_2} \in I_2$.

Because I_2 contains infinitely many terms of the sequence by our choice. This will always satisfy this.

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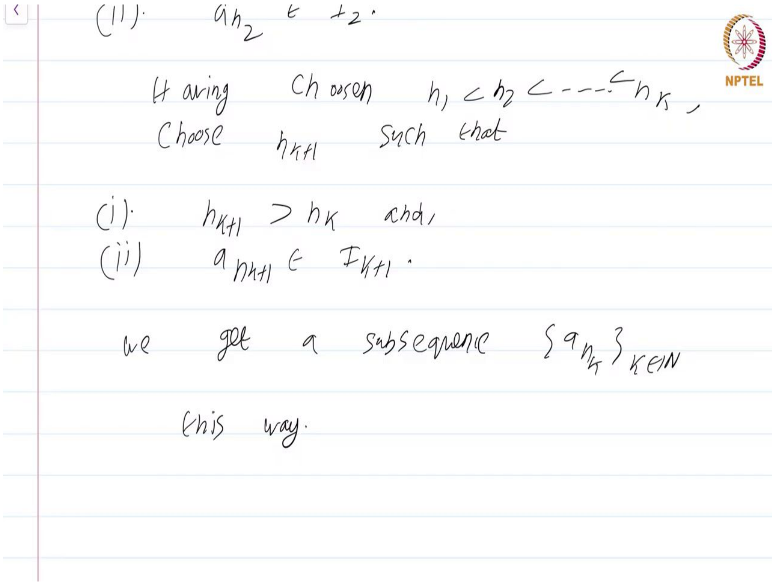

Having chosen $n_1 < n_2 < . < n_k$, choose n_{k+1} such that

(i). $n_{k+1} > n_k$ and (ii) $a_{n_{k+1}} \in I_{k+1}$.

So, you can see where this is going. We are trying to construct a subsequence such that the n_k^{th} term is present in I_k. That is the basic idea.

The first term we do not really care we are just setting n_1 to be 1. For all the other terms we are making sure that a_{n_k} is there in I_k. Now, we get a subsequence, a_{n_k}, $k \in \mathbb{N}$, this way .

Now, as you can guess this subsequence is not any old subsequence it actually converges.

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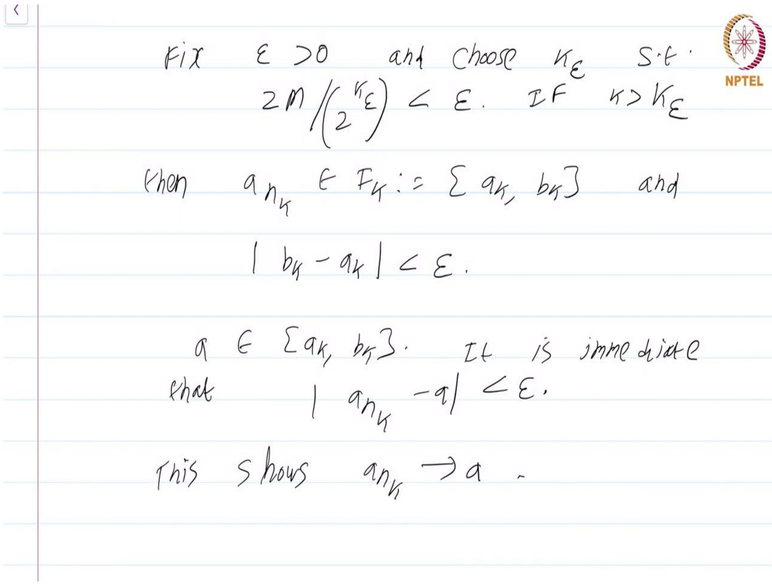

How do you show that converges? Well, fix $\epsilon > 0$ and choose K_ϵ such that $2m/2^{K_\epsilon} < \epsilon$. Just choose K_ϵ so large that $2m/2^{K_\epsilon} < \epsilon$. This can always be done .

If $k > K_\epsilon$ then $a_{n_k} \in I_k$ by our choice which is actually; we will just call it $[a_k, b_k]$ and $|b_k - a_k| < \epsilon$. Note this follows because at each stage the interval I_k is obtained from I_{k-1} by taking either the right half or the left half.

That means, $b_k - a_k$ at each stage is half the size of $b_{k-1} - a_{k-1}$. That is I_k is half the size of I_{k-1}. We started off with 2m; size 2m because we started off with [-m.m]. So, it will follow that if you make the choice of $k > K_\epsilon$ being chosen such that $2m/2^{K_\epsilon} < \epsilon$ then it will follow that $|b_k - a_k| < \epsilon$

And notice also that $a \in [a_k, b_k]$. It is immediate that $|a_{n_k} - a| < \epsilon$ because a is also following in this interval $[a_k, b_k]$.

So, this shows $a_{n_k} \to a$. So, we have found the required subsequence a_{n_k} by subsequently partitioning the interval [-m,m] in such a manner that at each stage you are always left with a interval which contains infinitely many indices for which a_n is there in that interval and from these intervals you are choosing one term in such a way to get a subsequence.

And because these intervals are shrinking by 2, at every stage it's becoming half, it is easy to show that the intersection will have one point and that point will be a limit of this constructed subsequence a_{n_k}; so this concludes the proof.

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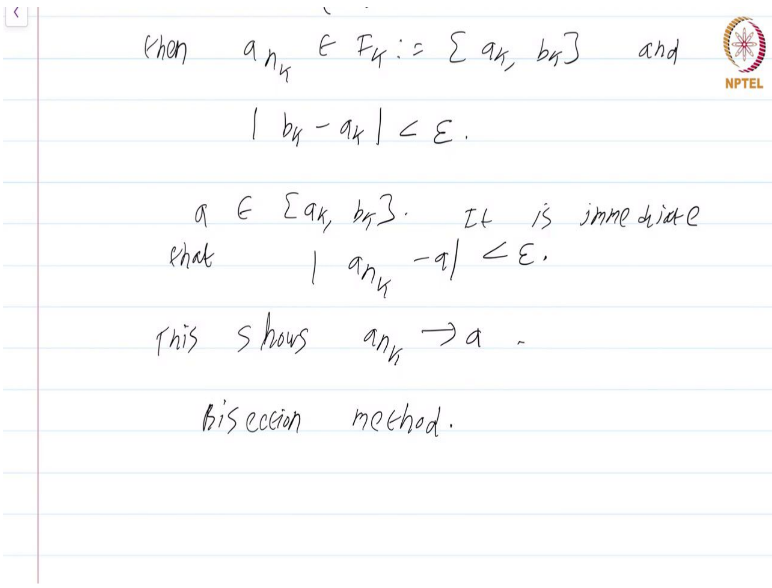

This is the more classical proof and this method is called the bisection method. The method that we have adopted here, it is called the bisection method. At each stage we are bisecting the intervals by 2. So, the second proof is a bit long, but there are no tricks involved, it is a straightforward proof. It is just a bit technical. So, please study this proof carefully.

This is a course on real analysis and you have just watched the module on the Bolzano Weierstrass theorem.