Real Analysis - I

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Lecture – 10.1
Monotone Sequences

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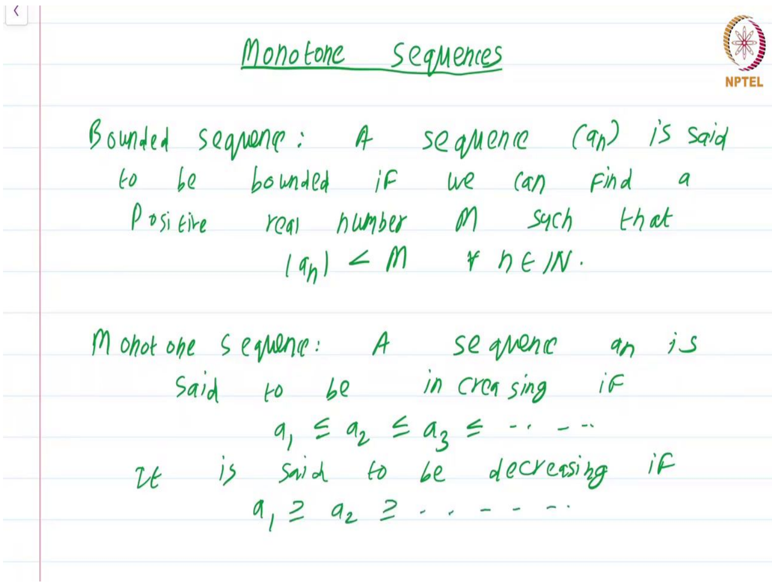

In this module, we shall see a very convenient criteria for checking whether certain type of sequences are convergent without having any candidate idea of what the limit is going to be. First, let me begin with two definitions. The first is that of a bounded sequence.

A sequence a_n is said to be bounded if we can find a positive real number M such that $|a_n| < M \quad \forall n \in \mathbb{N}$.

Every term in the sequence is bounded in absolute value by the number M.

The second definition is monotone sequence.

A sequence a_n is said to be increasing if $a_1 \le a_2 \le a_3 \le ..$. It is said to be decreasing, if $a_1 \ge a_2 \ge ..$.

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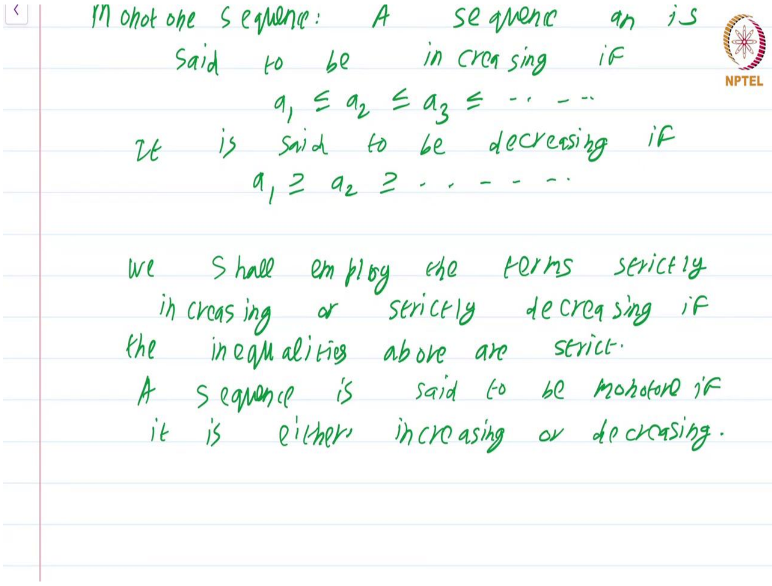

We shall employ the terms strictly increasing or strictly decreasing, if the inequalities above are strict.

A sequence is said to be monotone if it is either increasing or decreasing.

Of course, we shall also sometimes employ the term strictly monotone to suggest that the terms in the sequence are strictly greater, I mean $a_1 < a_2 < a3 < ....$.

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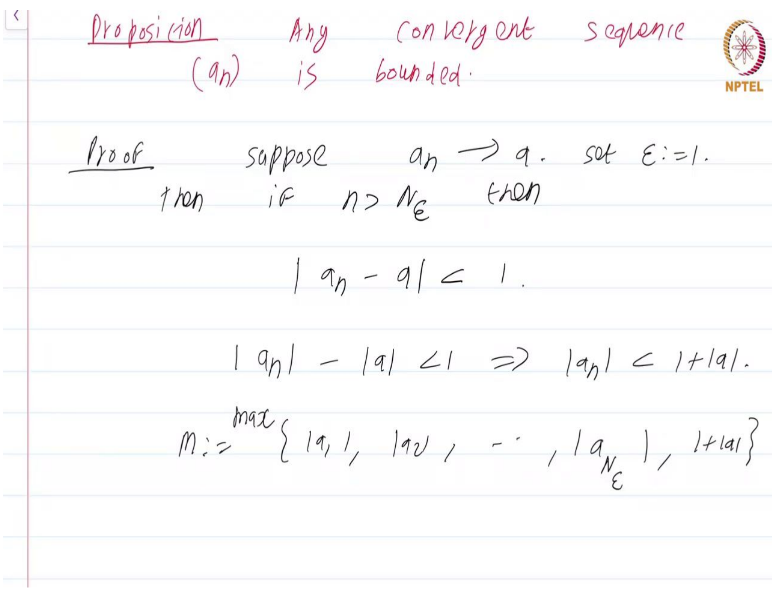

So, now we have a very simple proposition about convergent sequences.

Any convergent sequence a_n is bounded

Proof: Suppose, $a_n \to a$. Since the sequence is convergent, it must have some limit, let us say a. Set $\epsilon := 1$.

Then, if $n > N_\epsilon$, then $|a_n - a| < 1|$, that is the very definition of this N_ϵ. That is whenever the number $n > N_\epsilon$, the terms of the sequence a_n must be at the max of ϵ close to the limit; here epsilon is 1.

But by the reverse triangle inequality, this is just saying $|a_n| - |a| < 1$. In other words, $|a_n| < 1 + |a|$. So, it seems like we have already satisfied the definition of bounded, but not quite. This final inequality $|a_n| < 1 + |a|$ is true only for those terms of the sequence that are beyond N_ϵ.

What you do is you set $m := \max |a_1|, |a_2|, ..., |a_{N_\epsilon}|, 1 + |a|$.

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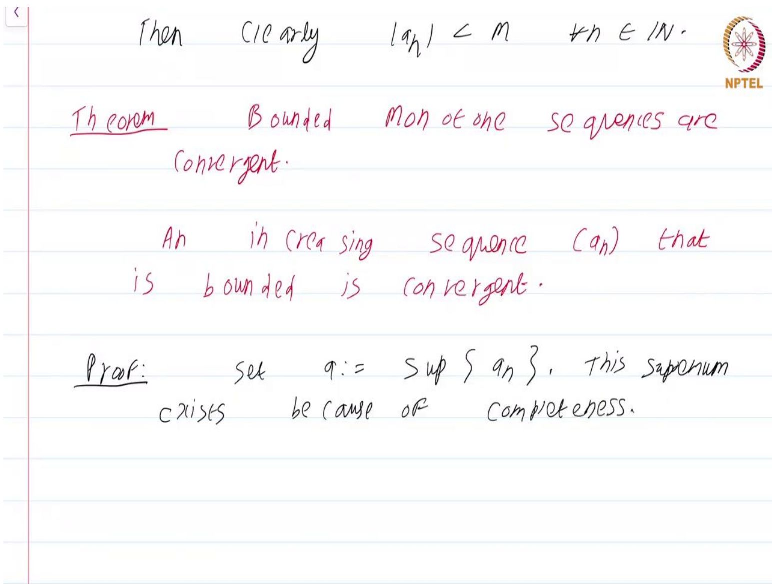

Then $|a_n| < M \quad \forall n \in \mathbb{N}$. That is how this M has been constructed and we are done with the proof

Now, we come to the important theorem that says bounded monotone sequences are convergent and the increasing sequence a_n that is bounded is convergent.

Now, the motivation for studying this monotone sequences is the previous proposition which says that convergent sequences must be bounded. So, our guess would be that bounded sequences are also convergent, but that is clearly not true.

Think of a counter example yourself. The next best thing is to say that at least monotone sequences that are bounded will be convergent. Now, I have here stated it for increasing sequence; think about how you can prove the same thing for decreasing sequences.

Proof: To show that a limit exists, we need some candidate. Now, what is happening? This sequence is increasing, but it cannot increase beyond a particular point M simply because the sequence is bounded.

So, what we will do is we will set $a := \sup a_n$, we are putting all the a_n's as a set and we are taking the supremum and this supremum exists. Why? Because of completeness. Now, our candidate for the limit is obviously going to be the supremum. It is the least element such that no element in the sequence (a_n) exceeds it. So, it is a very nice candidate for the choice of limit. Fix $\epsilon > 0$.

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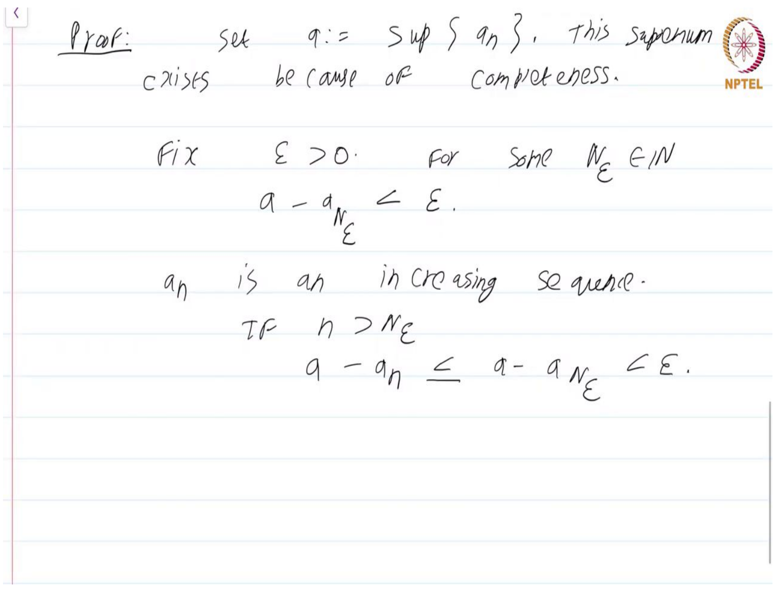

Now, recall the equivalent characterization of supremum as that for each such ϵ, there is always an element in the set which is ϵ close to the supremum. In other words, for some N_ϵ in the natural numbers $a - a_{N_\epsilon} < \epsilon$. Because this is the supremum, there is an element in the set which we are calling a_{N_ϵ} such that $a - a_{N_\epsilon} < \epsilon$.

But a_n is an increasing sequence.. What will that tell us? That will tell us that if $n > N_\epsilon$, $a - a_n < a - a_{N_\epsilon} < \epsilon$ because a_n is an increasing sequence. In fact, to be 100 percent precise, I must write less than or equal to because it could be the case that the sequence becomes sort of constant after a while; but this is less than ϵ .

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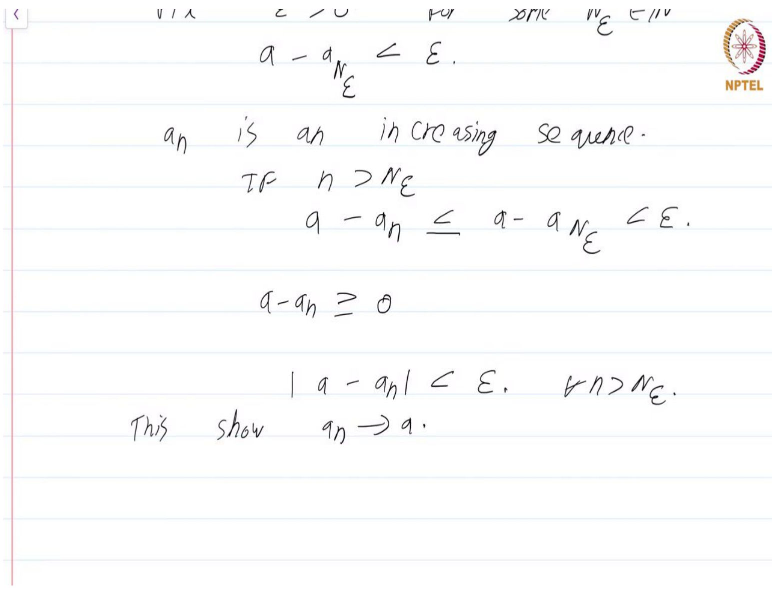

We already know that $a - a_n \geq 0$. How do we know this? Because a is the supremum of the set of a_n 's. In other words, we have shown that $|a - a_n| < \epsilon$, $\forall n > N_\epsilon$.

This is just a fancy way of saying that we have shown that the definition of convergence is satisfied for a_n with the limit a right. For each ϵ, we have shown that $|a - a_n| < \epsilon$, $\forall n > N_\epsilon$.

. This shows a n converges to a ok.

Please prove the other case when a_n is not an increasing sequence, but rather a decreasing sequence. This is a course on real analysis and you have just watched the module on monotone sequences.