Real Analysis - I
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Lecture – 9.3
Some Special Sequences

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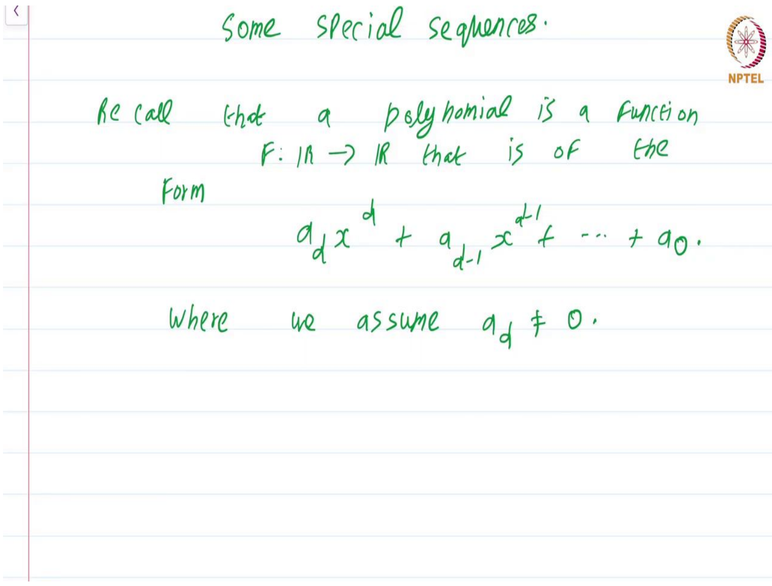

In this module, we will see some more examples of sequences and convergence, but this time we will study Some Special Sequences that are of great importance throughout your studies in analysis. So, let me recall a definition;

Definition: A polynomial is a function $F : \mathbb{R} \longrightarrow \mathbb{R}$ that is of the form, $a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0$. And here for definiteness, where we assume $a_d \neq 0$ and this d is called the degree of the polynomial.

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1. Let $a > 1$ be a real number. Then

$$\lim_{n \to \infty} \frac{a^n}{n} = +\infty.$$

write $a = 1 + b$, $b > 0$.

$$\frac{(1+b)^n}{n} = \frac{1 + nb + \frac{n(n-1)}{2} b^2 + \cdots}{n}$$

$$\geq 1 + b + \frac{(n-1)b^2}{2} \geq \frac{(n-1)b^2}{2} \to +\infty$$

This shows the claim.

Now, I am going to do some studies of sequences that involve polynomials . So, first example; you might have heard of this colloquial expression that an exponential grows faster than any polynomial growth. So, let us see how to translate this in terms of sequences.

Let a > 1 be a real number, then $\lim_{n \to \infty} \frac{a^n}{n} = +\infty$.

In other words, this sequence $\frac{a^n}{n}$ diverges to +infinity. That means the growth of a^n cannot be counterbalanced by the growth of n; a^n sort of grows much faster than just n. How do you show this? Well, again we are going to use the binomial theorem; what we do is. Write a = 1 + b, where b > 0 .

Consider $\frac{(1+b)^n}{n}$; this you can see is actually 1 + nb; now I need to write one other term because there is a n in the denominator. So, I would need to go further along in the binomial expansion, then what was needed the last time when we just considered the sequence a^n with no n in the denominator ; + some other terms. I do not care about what they are divided by n.

Now, why did I consider till the level 2? Well, simply because this I can write is greater than or equal to $1 + b + (n-1)b^2$. I forgot a by 2 here, the binomial coefficient will actually have by 2. So, let me just write that down by n; so $\frac{(n-1)}{2}b^2$.

So, I am ignoring all the other terms and since I mean I am allowed to do all this because every single term here is positive and this is certainly going to be greater than or equal to $\frac{(n-1)}{2}b^2$. And this last thing converges or rather diverges to +infinity . So, this shows the claim.

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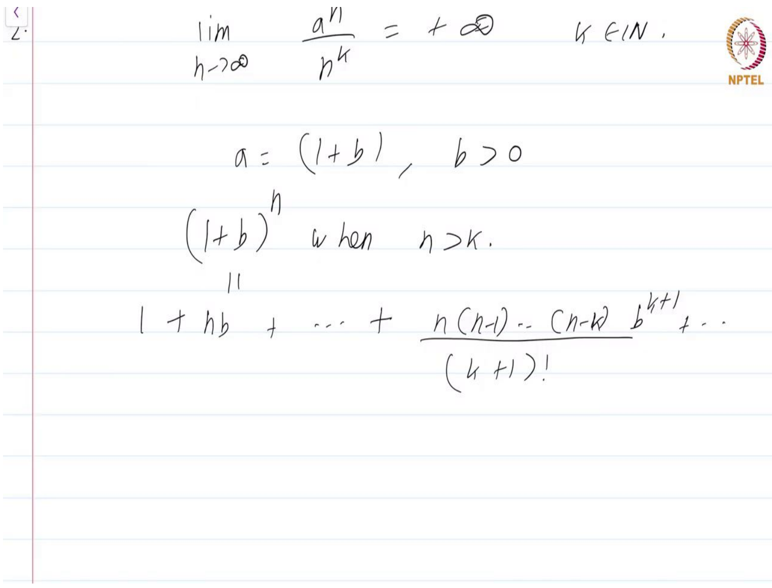

So, we have seen that a^n goes faster than n but originally. I made the remark that a^n goes to infinity faster than any polynomial. How do you deal with that? Well, that is the second example which is one layer even more complicated than what we have seen. That is, $\lim_{n\to\infty} \frac{a^n}{n^k}$, where k is a natural number .

So, this time we are dividing by any higher order monomial. Now, how do I show this? Well, this is slightly trickier, but it is exactly the same idea. Write a as, again a > 1, write a = 1 + b and b > 0 and look at $(1+b)^n$. Look at $(1+b)^n$, when n > k..

Now, how will the binomial expansion look like? This is nothing, but $1 + nb + \cdots$ some other terms and I focus on one particular term and you should be able to guess what this particular term is. It is that term which has $\frac{b^{k+1}}{(k+1)!} +$ some other terms.

So, I focus on this particular term which has b^{k+1} in it. So, this is what $(1+b)^n$ will be; when n > k. Now I have to divide this by n^k.

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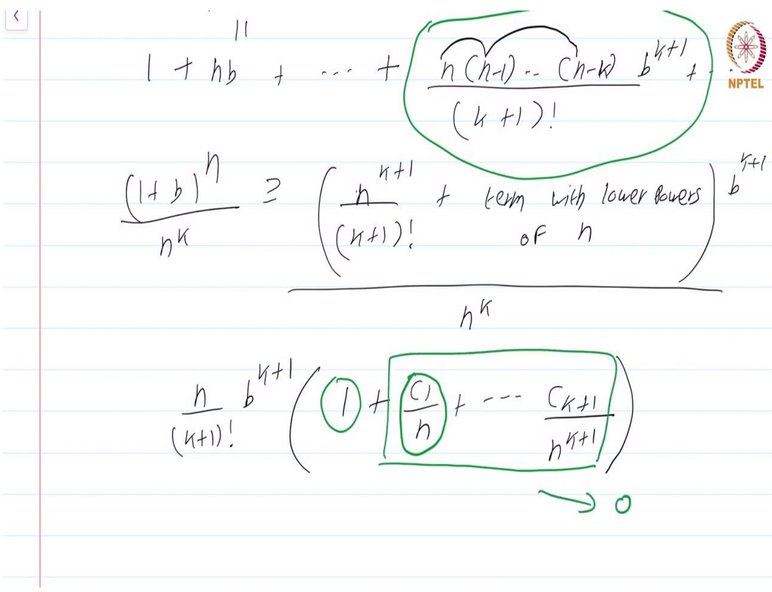

I want to consider this and I want to show that it is greater than or equal to some quantity that sort of goes to +infinity. Now, how do I do that? Well, observe that every term here is also positive. So, I can ignore all the terms; except the term that I am really interested in which is this. I can ignore all the other terms.

Now, if you look at this term and expand out $n(n-1)\cdots(n-k)$. It is easy to see that this whole thing will be greater than or equal to some n^{k+1};

$$\frac{\left(\frac{n^{k+1}}{(k+1)!} + \ terms\ with\ lower\ powers\ of\ n\right)b^{k+1}}{n^k}.$$

So, what I have done is; I have focused on one of the terms, the one having b^{k+1}, as the coefficient. Then, what I am doing is this binomial coefficient that comes. I am expanding that and I notice that there will be one term which has k power 1; that is one when you multiply all the ends together. There are k + 1 of them and then oh sorry; I made a mistake here, this is n^k.

So, when you do this division; it is easy to see that what you will be left out with n^k is $\frac{n}{(k+1)!}b^{k+1}$ that will be the first term. Then, there are all lower order terms whose coefficients could be very complicated, but I don't really care about the coefficients. All I care about is; how do they look like and you can see that I can just take everything outside and it will look like $\frac{c_1}{n} + \cdots + \frac{c_{k+1}}{n^{k+1}}$.

What have I done? Well, there is an n^k coming in the denominator and that will cancel off with n^{k+1} and you will be left with an $\frac{n}{(k+1)!}b^{k+1}$. Now, the next term will involve n^k and when you divide by n^k that will be gotten rid of. But, I am still taking an n outside; so you will be left with some coefficient $\frac{c_1}{n}$.

So, originally this term add n^k that got cancelled out, but there is an n in the denominator because I want to keep one n outside. You will understand in a moment why I want to do that. And similarly, the final term will be a constant term out of which I am dividing an n^k and I am taking an n outside. So there is an n^{k+1}. And this one accounts for the very first term which is this $\frac{n}{(k+1)!}b^{k+1}$.

Now, why did I do this? Well, I did this because now I can just say all of these terms; irrespective of what the coefficients c_1 to c_{k+1} are all these terms go to 0. All of this converges to 0. So, I am just left with $\frac{n}{(k+1)!}b^{k+1}$ and this clearly goes to + infinity, diverges to +infinity.

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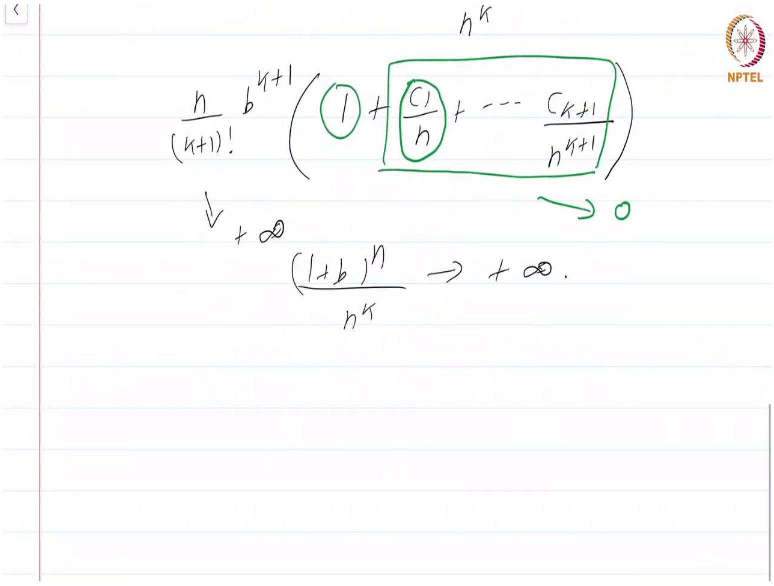

So, what we have shown is that $\frac{(1+b)^n}{n^k}$ is in fact, greater than or equal to a quantity that goes to +infinity. So, $\frac{(1+b)^n}{n^k}$ diverges to +infinity as required. So even when you take an exponential and divide by a higher degree term, you still get divergence to +infinity.

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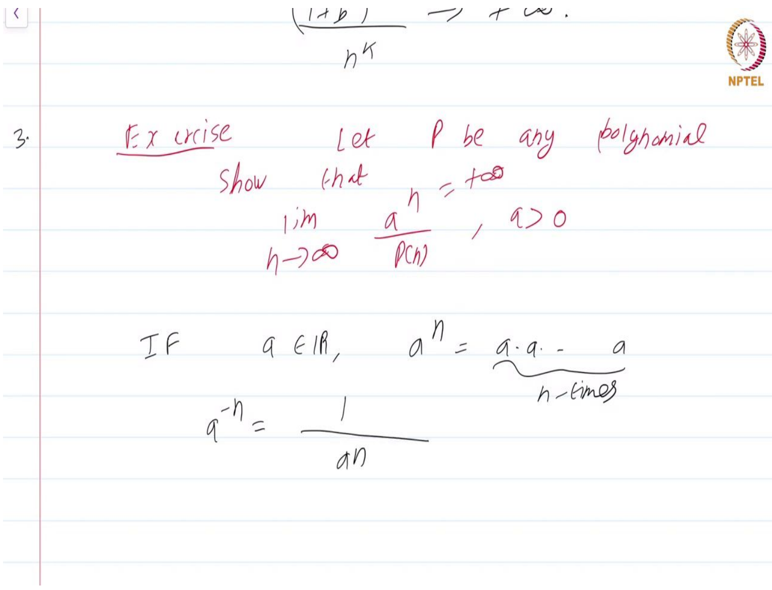

Now, Example 3 as you can guess and it is going to be an exercise for you;

Exercise: let P be any polynomial, show that $\lim_{n\to\infty} \frac{a^n}{P(n)} = \infty,\ a > 0$,as usual, excellent.

So, now we have seen some examples involving polynomials and exponentials. Now let us see some more sophisticated examples involving exponentials. Before that, let me just make a remark;

if a is a real number, then we all know what a^n is; this is just $a \cdot a \cdot a \cdots a$, (n times) and $a^{-n} = \frac{1}{a^n}$. This is all valid when n is a natural number .

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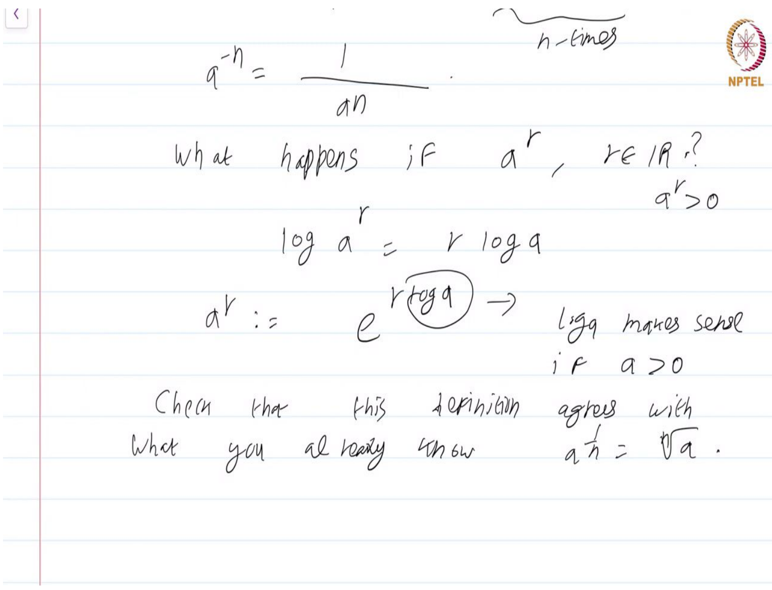

Now, what happens, if the power you take a^r, where r is a real number? Now, this can be defined in several ways, but the most elegant way is to do it via exponentials and logarithms. Now, how do you do that? Well, let us assume that you can take $\log a^r$. When can you do that? You can do that whenever $a^r > 0$, you can take $\log a^r$ and we know that this becomes $r \log a$.

So, we can just reverse engineer and define a^r to be exponential of $r \log a$ and this will make sense whenever $log a$ makes sense. In other words, whenever a > 0. Now, I am not yet defined what the exponential and logarithm are, we will do it later in the course. But I just want to make a remark that this is the definition of exponentials that we will take in this course. So, note that r could be anything; positive, negative, anything and check that this definition agrees with what you already know.

What do I mean by this? For instance $a^{\frac{1}{n}}$ should be nth root of a; for instance. So, please check this is the definition of exponential. So, now let me give some more examples of special sequences.

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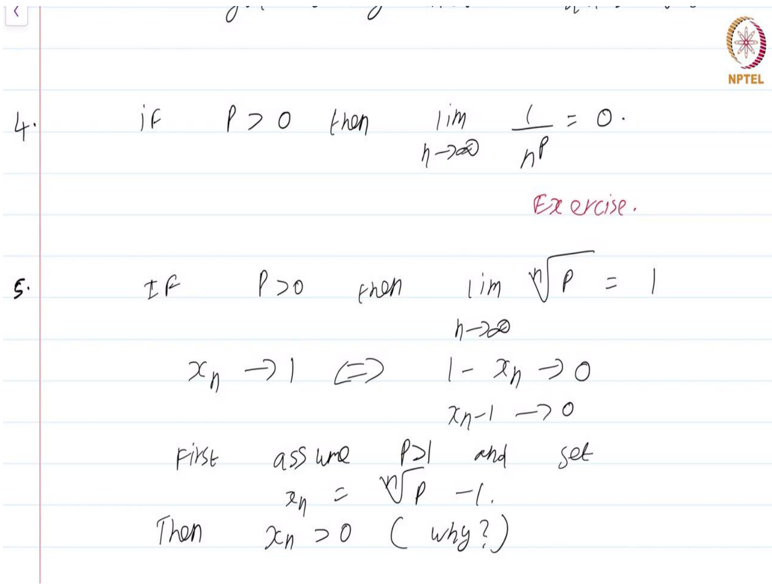

Example 4: If P > 0, then $\lim_{n\to\infty} \frac{1}{n^P} = 0$. And I am going to leave this as an exercise for you; this is not at all hard, we have already seen what happens when P = 1 and what happens when P = 2.

Similar, remarks will apply even for this P being any real number greater than 0 . So, that is one more example that I want you to do as an exercise.

Example 5. If P > 0, then $\lim_{n\to\infty} \sqrt[n]{P} = 1$. When you take a positive number; real number and iteratively keep taking the nth roots, you get closer and closer to 1.

Now, how are you going to show this? Well, what you do is; you do a particular trick. Suppose you want to show that $x_n \to 1$, then one strategy that you can adopt is to see that this is same as showing that $1 - x_n \to 0$. It is exactly the same thing in a different form .

So, now, what I am going to do is; of course, I can also say $x_n - 1 \to 0$. You can do one of these also; instead of showing $x_n \to 1$, you can show that $x_n - 1 \to 0$.

So, what I am going to do is first , assume P > 1 and set $x_n = \sqrt[n]{P} - 1$. Then, then $x_n > 0$; why? Please, think about why this is true . Now, I want to show that this; a quantity $x_n \to 0$.

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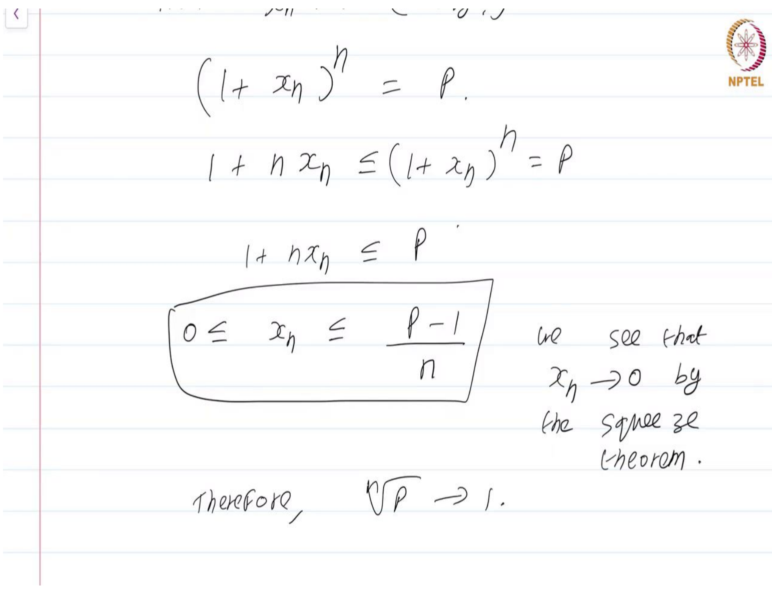

Now, what do I do, how do I approach this? Well, observe that $(1+x_n)^n = P$. Again, by the binomial theorem; we can write $1 + nx_n \le (1+x_n)^n = P$. In other words, $1 + nx_n \le P$ right or $x_n \le \dfrac{P-1}{n}$.

We already know that $0 \le x_n$. Combining these two, we see that $x_n \to 0$ by the squeeze theorem. So, we were able to prove that $x_n \to 0$ and therefore, $\sqrt[n]{P} \to 1$. So, again this is very similar to this proofs we have seen earlier, we just use the binomial theorem .

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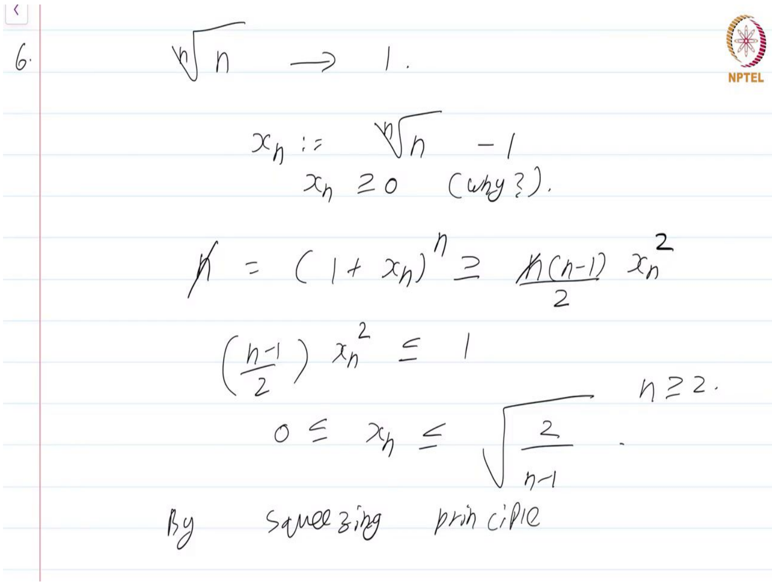

Example 6: We do the same thing that we have been doing, except with the more complicated twist; $\sqrt[n]{n} \to 1$. Again to tackle this, we will transform the sequence into a sequence that we would want to converge to 0 by setting $x_n = \sqrt[n]{n-1}$. Again $x_n \geq 0$ and I want you to think about why this is true ?

Now, $n = (1 + x_n)^n$ and now we expand this by binomial theorem. We will have many terms; I am interested in only one of them. This will be $\geq \frac{n(n-1)}{2} x_n^2$. Now, since n is greater than or equal to this; we can cancel off the n. We will get $\frac{n-1}{2} x_n^2 \leq 1$. In other words, $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$. This will be valid whenever $n \geq 2$.

We have to be a bit careful because when n = 1, the denominator becomes 0 and this quotient does not make any sense. So, again by squeezing principle or squeeze theorem, $x_n \to 0$. Hence the $\sqrt[n]{n} \to 1$.

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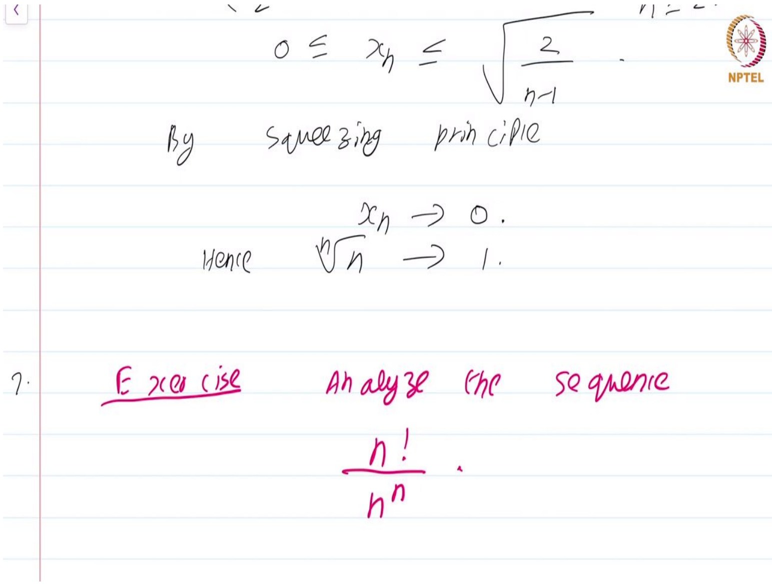

So, one final example which is in fact, I am going to leave it to you as an exercise .

Analyze the sequence $\frac{n!}{n^n}$; this is a somewhat involved example; please work this out in great detail .

So, this concludes this module on Some Special Sequences and you are watching this course on Real Analysis.