Real Analysis - I

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Lecture – 9.2
Examples of Convergent and Divergent Sequences

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In this module we are going to see several examples, this is in some sense the most important example as the popular saying goes an example is worth a thousand proofs.

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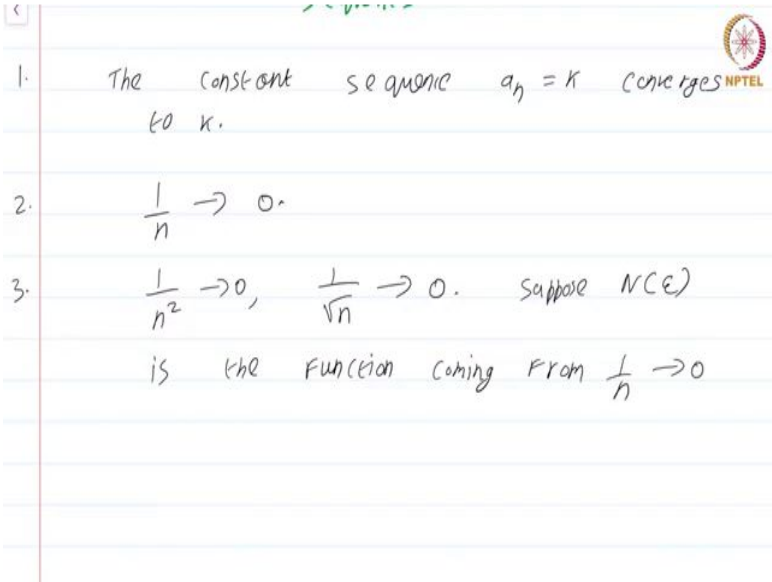

So, let us start the first example, the constant sequence $a_n = k$ converges to k. Well there is really nothing to prove for the choice of $N(\epsilon)$ you can choose any natural number this will work .

The second one we have already seen $\frac{1}{n} \to 0$, this is an example that we saw in some detail.

3. $\frac{1}{n^2} \to 0$ $\frac{1}{\sqrt{n}} \to 0$.

Now how do you show this? Well. There are two ways, one of them is the clever way, the other one is the way that you should not even be considering. Let us do it the clever way , clever way is as follows.

Suppose $N(\epsilon)$ is the function coming from $\frac{1}{n} \to 0$. Since $\frac{1}{n} \to 0$, there is a function $N(\epsilon)$ that does what is needed in the definition of convergence.

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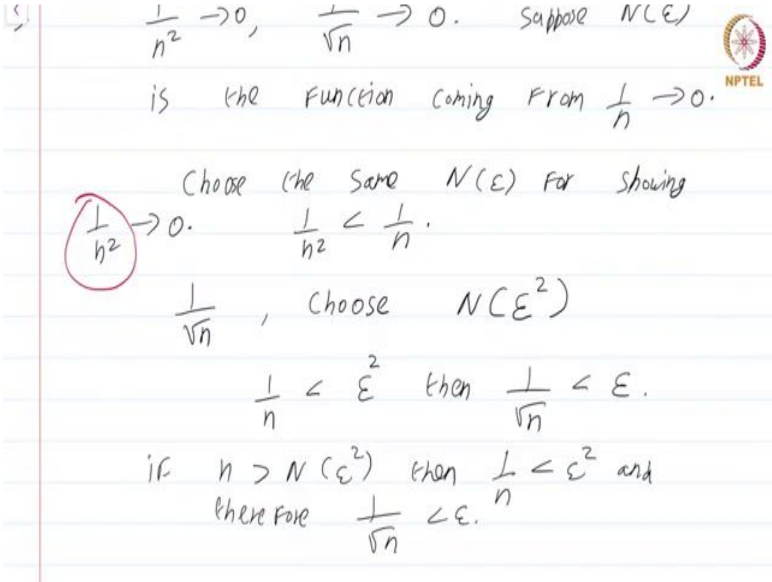

Now, choose the same $N(\epsilon)$ for showing $\frac{1}{n} \to 0$, the same will work. why will the same work? Because $\frac{1}{n^2} < \frac{1}{n}$, that is the reason. So, the same $N(\epsilon)$ will work.

What about for $\frac{1}{\sqrt{n}}$? Well just choose $N(\epsilon^2)$. How does this l if $\frac{1}{n} < \epsilon^2$ then $\frac{1}{\sqrt{n}} < \epsilon$.

So, if $n > N(\epsilon^2)$ then $\frac{1}{n} < \epsilon^2$ and therefore, $\frac{1}{\sqrt{n}} < \epsilon$. So, this quickly dispenses of $\frac{1}{\sqrt{n}}$ also. Just a remark for this sequence $(\frac{1}{n^2})$, we could have also used the fact that $\frac{1}{n} \to 0$ and applied the fact that product of limits product of sequences the limit of a product of a sequence is the product of the limits we could have done that also.

So, these are some easy examples, now let us start seeing some more slightly complicated ones .

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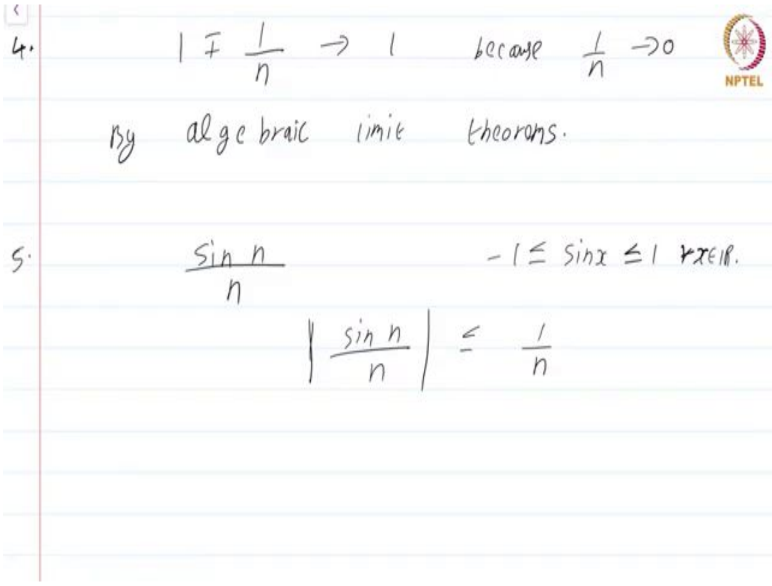

4. $1 \mp \frac{1}{n} \to 1$. This sequence is why is that the case? Because $\frac{1}{n} \to 0$ and we can use the algebraic limit theorems. So, the sequence $1 \mp \frac{1}{n} \to 1$. This is not really complicated. I did not keep up my promise, no worries.

Look at this one $\frac{\sin n}{n}$. Now this looks complicated, but at first sight you might think well. Why not just apply the fact that quotients if you have $\frac{a_n}{b_n}$ then the limit is just $\frac{a}{b}$, where $a_n \to a$ and $b_n \to b$. Well this is slightly problematic; because the denominator sequence (n) certainly does not converge it diverges to infinity and the numerator sequence look at $(\sin n)$: $\sin 1, \sin 2, \sin 3, \sin 4, \ldots$ and so on.

If you were to plot this you will notice that this oscillates wildly. So, it does not really clear what I am supposed to do with this. Well let us manipulate first . Let us look at $|\frac{\sin n}{n}|$ because at the end of the day if this were to have a limit I would guess it is 0 because the denominator has the term n and $\frac{1}{n} \to 0$. I would guess that this also converges to 0. Look at $|\frac{\sin n}{n}|$.

Now, all of us are familiar with the fact that $-1 \leq \sin x \leq 1 \ \forall x \in \mathbb{R}$. This comes from the definitions. I mean the various basic properties of the sine function. We will rigorously define what the sine function, the cosine function, the logarithm exponential in a later module. But for the time being you can just take this for granted. You have known this all the way back from 8th or 9th standard.

So, $-1 \leq \sin x \leq 1 \ \forall x \in \mathbb{R}$. So, certainly this is going to be less than or equal to $\frac{1}{n}$.

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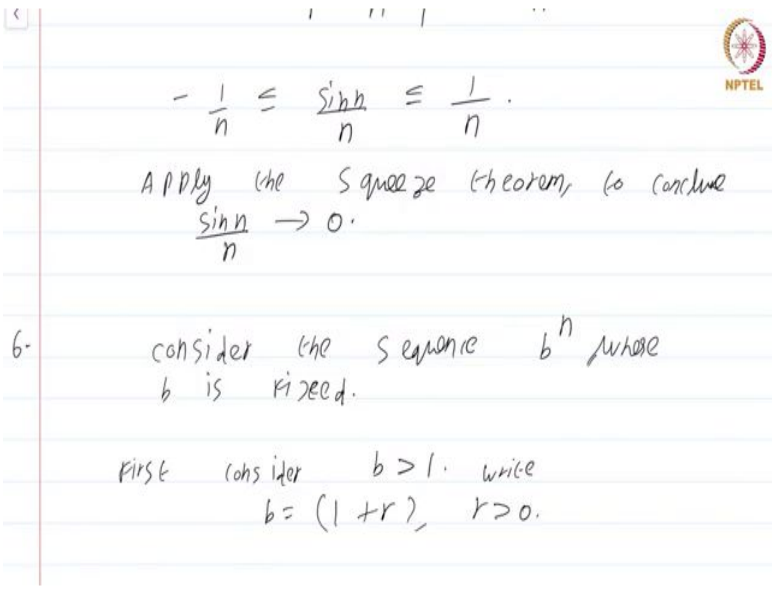

Well in fact, what is this saying , this is just saying $\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$. Now, apply the squeeze theorem . Both the RHS sequence $\frac{1}{n} \to 0$, the LHS sequence $\frac{1}{n} \to 0$. Therefore, $\frac{\sin n}{n}$ has to converge to 0. Apply the squeeze theorem to conclude $\frac{\sin n}{n} \to 0$. So, this was a somewhat involved example.

Now let us see somehow, what I am going to do now is consider the sequence (a^n) . Note, this is not a_n; a^n. So, what I am going to do is to make it ultra precise. Consider the sequence b^n, where b is fixed. Now what happens to this sequence? Well, it depends on whether the $|b| > 1$ or $|b| \ngtr 1$, that is it lies between -1 and 1 .

To see precisely what happens, what we will do is. First we will first consider b > 1. The case b = 1 is the simplest, it is just going to be 1, 1, 1, 1, 1,..... it converges to 1. Consider the case b > 1. What I do is; write b = 1 + r, where r > 0. I can do this because I am assuming b > 1 .

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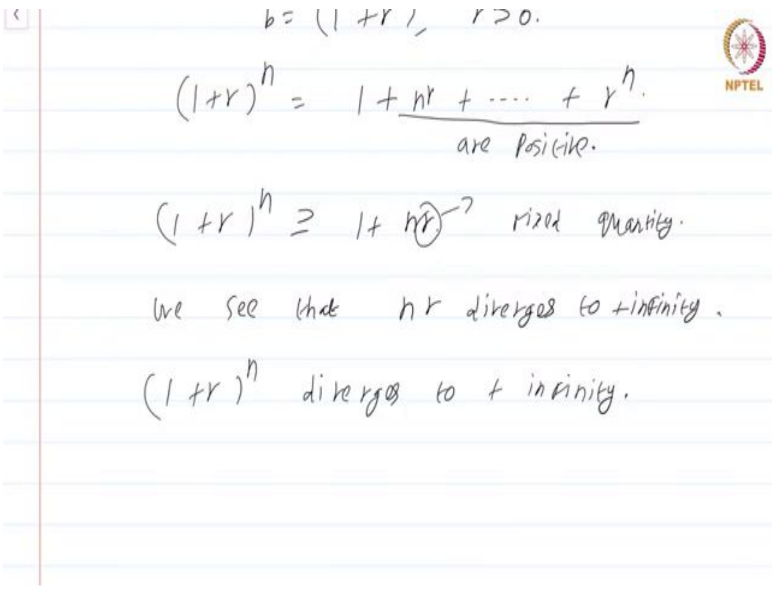

Now, I want to compute $(1+r)^n$, that is the nth term of the sequence. How do you compute this well? We know the familiar binomial formula. If you are not familiar with this just check your high school textbooks again you would have proved it using induction or by a combinatorial argument . Now what is this? This is nothing, but $1 + nr + \cdot + \cdot + \cdot + r^n$. I am not really interested in the in between terms they will involve the various and choose something. I am not really interested in those terms.

But the key fact is all these terms are positive. So, in any case, $(1+r)^n \geq 1 + nr$. I am just ignoring all the other positive terms. So, $(1+r)^n \geq 1 + nr$. But now r is a fixed quantity. That means, when I multiply n with r, we see that nr diverges to infinity. Please recall the definition of what diverges to infinity means; diverges to plus infinity to be ultra precise.

This just means that nr can be made larger than any pre- specified number if n is large enough. In fact, you can precisely write down how large n has to be. For $nr \geq M$ in terms of r that is easy to do . So, what this shows is $(1+r)^n$ also diverges to + infinity, . Now we have dealt with the case when this number b > 1.

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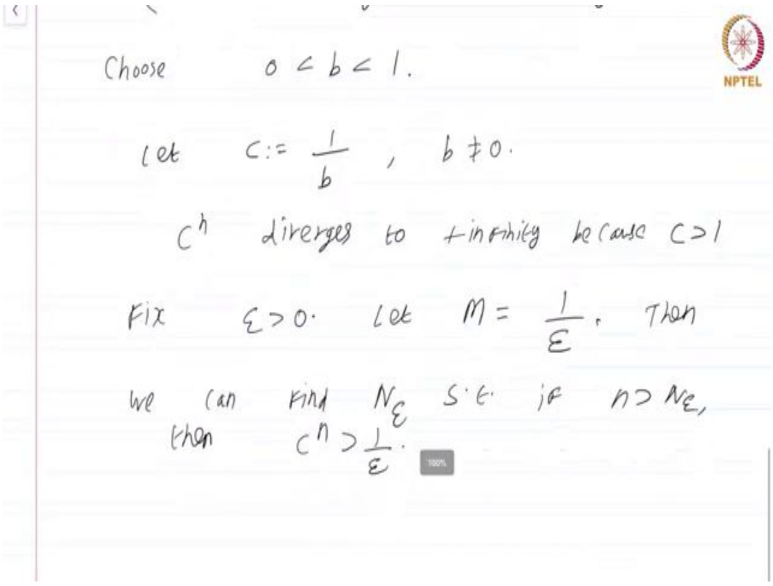

Now, choose b such that $0 \leq b < 1$. Now how do you deal with this case? You might be tempted to write b = 1 - r and take $(1-r)^n$. Well, don't do that, always try to use what has been shown so far. Let $c = \frac{1}{b}$. Let us see by definition $c = \frac{1}{b}$ well of course, if $b \neq 0$. Again the case b = 0 is really trivial. I am not going to leave that to you there is nothing really to show I will just take 0 < b < 1.

So, take $c = \frac{1}{b}$, now what happens? $\frac{1}{b^n}$ or in other words c^n diverges to +infinity, why? because c > 1 and that is the case we immediately just saw. Because c^n diverges to +infinity and $c^n = \frac{1}{b}$ this means that if n is chosen not n . Let me write down a full proof fix $\epsilon > 0$. Let $M = \frac{1}{\epsilon}$. Then we can find $N(\epsilon)$ such that if $n > N(\epsilon)$ then $c^n > \frac{1}{\epsilon}$.

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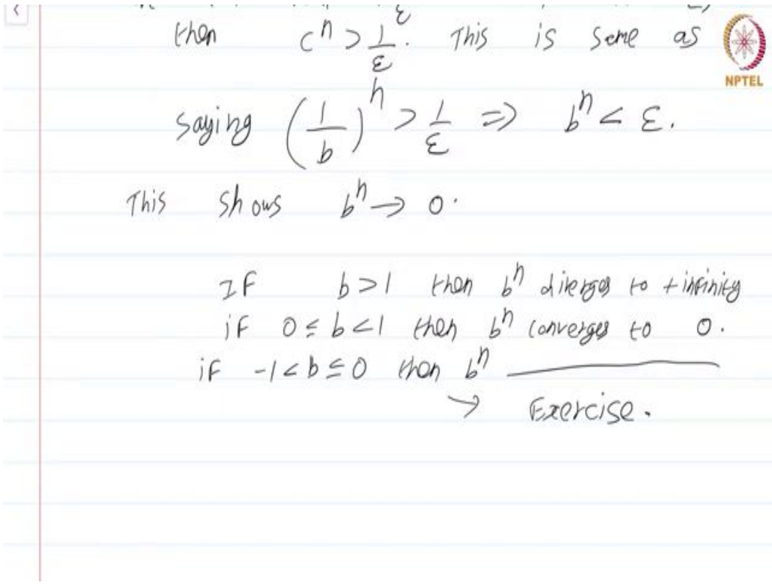

But this is same as saying $\frac{1}{b^n} > \frac{1}{\epsilon}$ which is same as saying $b^n < \epsilon$. So, this shows $b^n \to 0$. Let me summarize what is it that we have shown. We have shown that if b > 1 then b^n diverges to +infinity. If $0 \leq b < 1$ then $b^n \to 0$.

Now if $-1 < b \leq 0$ then b^n again converges to 0, why is this? Well, I am going to leave it to you. Exercise.

Excellent, this is a very interesting one. So, we have now shown that the sequence b^n, the behavior crucially depends on the choice of b.

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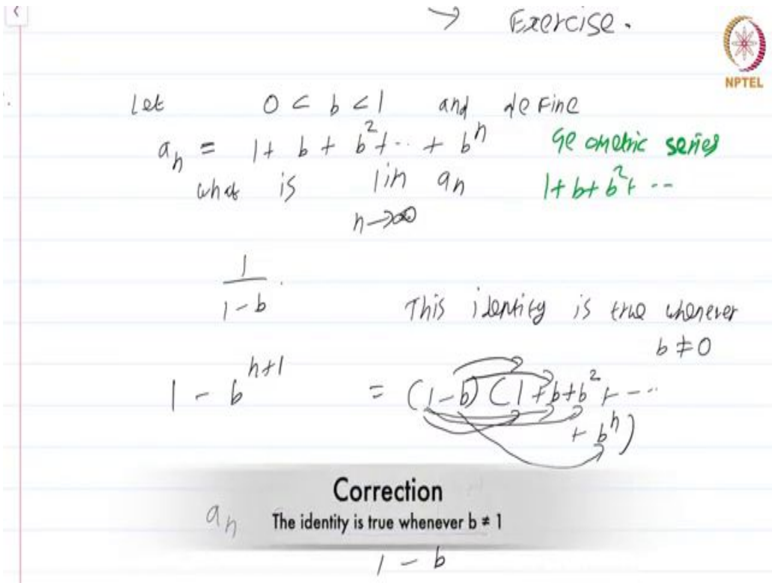

7. For this one let me choose a somewhat complicated example. Let me just deal with the case 0 < b < 1 and define $a_n = 1 + b + b^2 + \cdots + b^n$. I am defining the nth term of the sequence to be just $a_n = 1 + b + b^2 + \cdots + b^n$. What is limit of a_n as n goes to infinity? All of you know the answer: it is just $\frac{1}{1-b}$; this is called the geometric series, $1 + b + b^2 + \cdots +$ so on ,all the way up to infinity is called the geometric series.

So, this geometric series will serve as a motivation. In fact, this was related to what we saw in that story about chocolate wrappers. So, this series converges to $\frac{1}{1-b}$. How do you show that? Well, it is rather easy; you just have to use a bit of algebra. The bit of algebra is as follows, $1 + b + b^2 + \cdots + b^n$ I can write this as 1 - b into.

So, $(1 - b^{n+1}) = (1 - b)(1 + b + b^2 + \ldots + b^n)$.

In fact, this is all the way up to b^n, why is this correct? Well, there is a 1 accounted for by 1 into 1 then this $-b^{n+1}$ is accounted for by this product. What happens is all the intermediate terms b comes once as positive and once negative, b^2 comes once as positive and once negative, everything else just cancels off. So,

$(1-b^{n+1})=(1-b)(1+b+b^2+...+b^n)$. So, what we are interested in a^n is nothing, but $\frac{1-b^{n+1}}{1-b}$.

Now, if you were to observe this algebraic identity is true whenever $b \neq 0$, this is true. So, we have got $a_n = \frac{1-b^{n+1}}{1-b}$.

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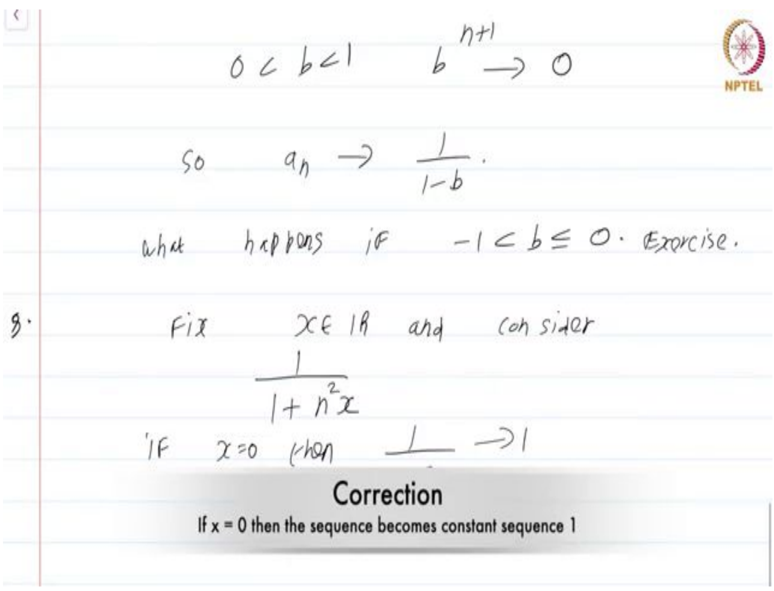

Excellent how does this help us. So, what the limit is well 0 < b < 1. So, $b^{n+1} \to 0$. That is what we just saw in the last example right. So, the numerator converges to 1 and the denominator is just the constant 1 - b. So, a_n converges to $\frac{1}{1-b}$, excellent. Now what happens $-1 < b \leq 0$. Again I leave it to as an exercise. Hint: you get exactly the same thing. So this shows that geometric series converge to $\frac{1}{1-b}$, excellent.

Now one more example before we conclude this set of examples. Again this is going to be a somewhat complicated one.

8. What I do is fix $x \in \mathbb{R}$ and consider $\frac{1}{1+n^2x}$.. What do I do with this? We want to show that if x = 0 then $\frac{1}{1+n^2x} \to 1$. There is nothing to show because this will just become $\frac{1}{n}$.

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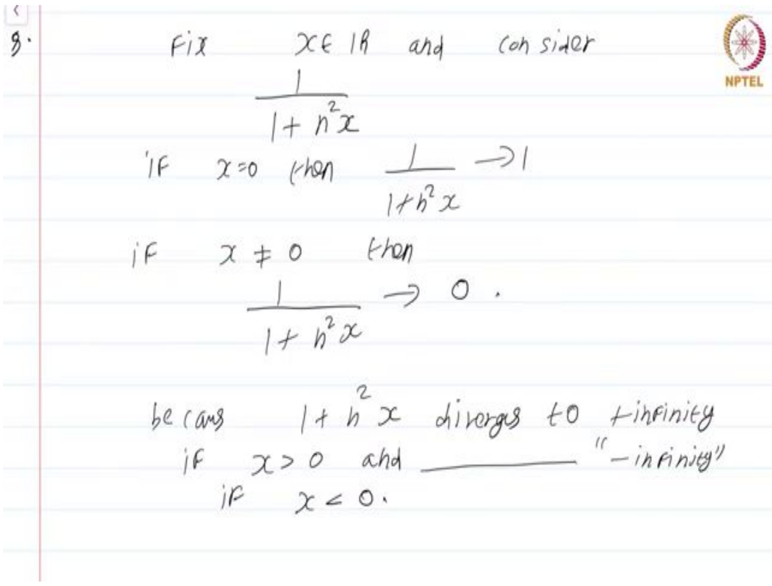

Now, if $x \neq 0$ then $\frac{1}{1+n^2x}$ converges to as you can guess 0. Why is this the case? Well, because $\frac{1}{1+n^2x}$ diverges to + infinity if x > 0 and well, I am not formally defined what diverges to -infinity is. But I let me just loosely use this diverges to -infinity if x < 0.

Because of this we immediately get that $\frac{1}{1+n^2x}$ must converge to 0 if x is any number other than 0 . So, this is one set of examples.

In the next module we will see some more set of examples that involve polynomials and exponentials together.

This is a course on Real Analysis; you have just seen the module on Examples of Convergent and Divergent Sequences.