Real Analysis - I

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Lecture – 9.1
Subsequences

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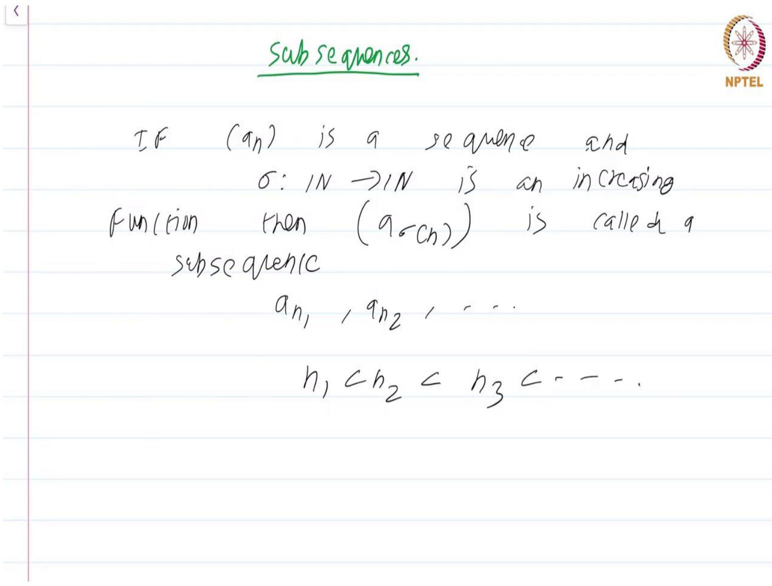

Let us talk a bit more about Subsequences in this module. First let me recall the definition of a subsequence.

If a_n is a sequence and $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ is an increasing function, then $a_{\sigma(n)}$ is called a subsequence. In other words, this is just a collection of terms from the sequence a_n in the same order. So it is $a_{n_1}, a_{n_2}, \cdots$, where $n_1 < n_2 < n_3, \cdots$. It is just terms of the sequence in the same order.

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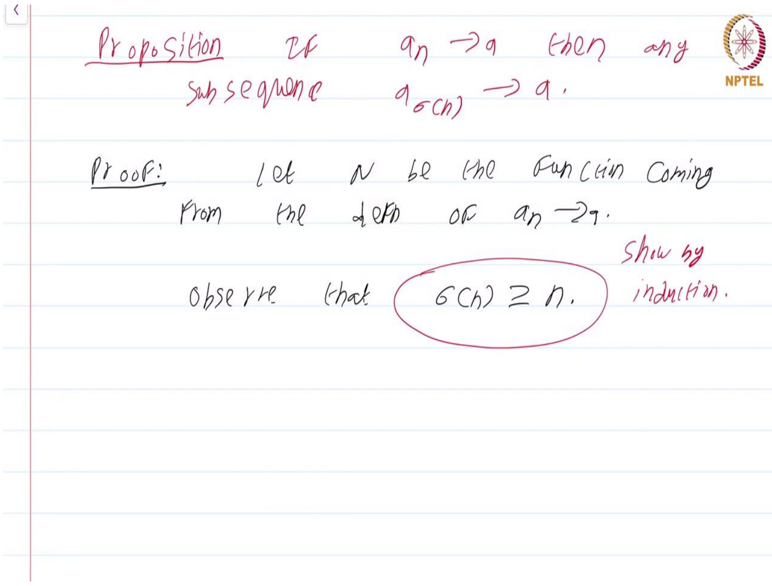

And a moment's thought should convince you that this proposition is true. If not, the proof certainty will; it is a very easy proposition.

If $a_n \to a$, then any subsequence $a_{\sigma(n)} \to a$. So, all subsequences of a convergent sequence converges to the same limit

Proof. Let N be the function coming from the definition of $a_n \to a$. Now, observe that $\sigma(n) \geq n$. Why is this? You can show this by induction . This is actually obvious to see show by induction. This just follows from the fact that σ is an increasing function.

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If $n > N(\epsilon)$ then

$|a_{\sigma(n)} - a| < \epsilon$

because $\sigma(n) \geq n > N(\epsilon)$

Proposition: Let (a_n) be a sequence such that each subsequence has a further subsequence that converges to a. Then $a_n \to a$.

Proof: Suppose $a_n \not\to a$. This means for some choice of $\epsilon > 0$ and

Now, if $n > N(\epsilon)$, then $|a_{\sigma(n)} - a| < \epsilon$ simply because $\sigma(n) \geq n > N(\epsilon)$. And that is it we are done. So, the same function N that works for $a_n \to a$ also works for $a_{\sigma(n)} \to a$. So, this was a fairly easy proposition.

The next one is a slightly more complicated proposition. It is sort of a converse for this proposition. Now, the naive converse of this is: "If every subsequence converges to the same limit a, then the sequence itself converges to the limit a".

Well, that is a bit naive and simplistic because (a_n) itself is a subsequence of the a_n. Therefore, saying that every subsequence converges to 'a' directly says that $a_n \to a$. It is not really much of a big deal.

We can do slightly better by stating a more nuanced version .

Let a n be a sequence such that each subsequence has a further subsequence that converges to 'a', then $a_n \to a$.

What this is saying is, no matter what subsequence you take, we are not guaranteed that that subsequence converges to 'a', but what we are guaranteed is that some subsequence of the subsequence you are considering converges to the point 'a'.

In this event, the whole sequence (a_n) itself converges to a. And the proof is by contradiction. You have to negate what it means.

Suppose, $a_n \not\to a$.

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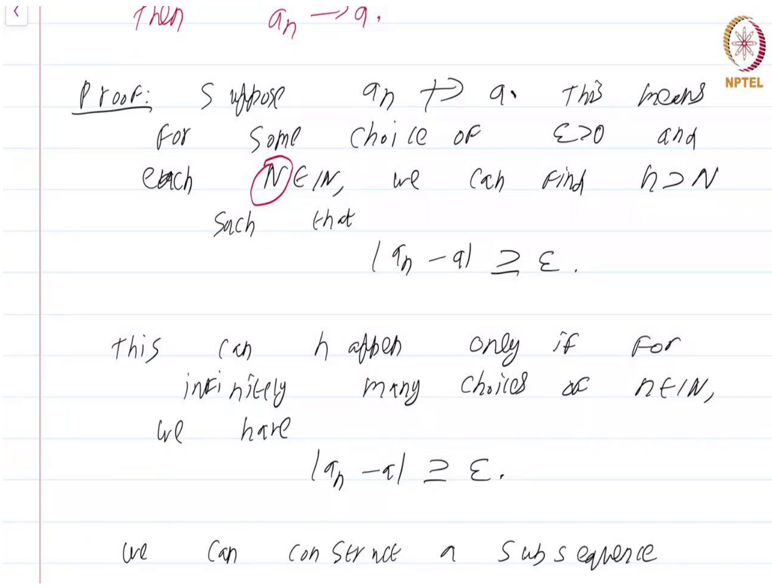

What does this mean? This means, for some choice of $\epsilon > 0$, and each $N \in \mathbb{N}$, we can find $n > N$ such that $|a_n - a| \geq \epsilon$.

What this is saying is, there is some ϵ that plays the role of an obstacle that cannot be surmounted. That means, no matter what N you choose, that will not work in the definition of convergence for this particular choice of ϵ. That means, $|a_n - a| \geq \epsilon$ for some $n > N$.

Now, I want you to prove that this can happen only if for infinitely many choices of $n \in N$, we have $|a_n - a| \geq \epsilon$.

Let me just give you a hint. Look at this N, and keep on increasing it. You will be able to get infinitely many terms n with $|a_n - a| \geq \epsilon$.

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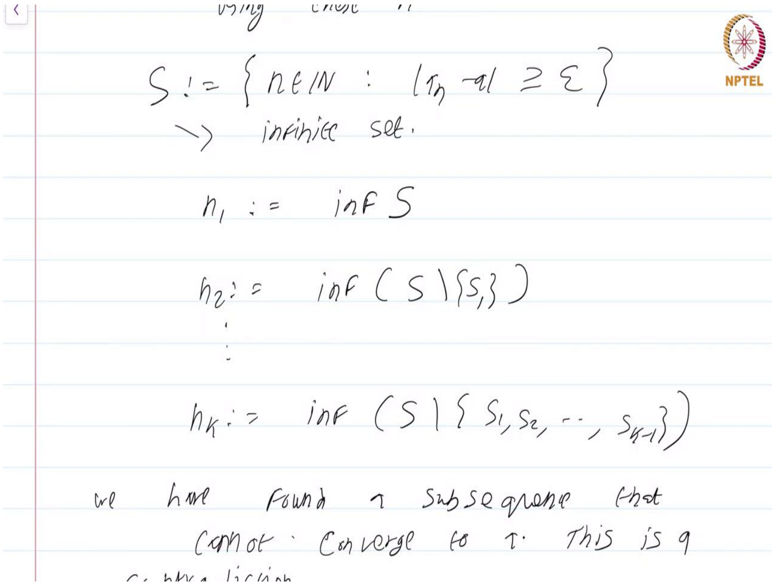

Now, it is fairly easy , we can construct a subsequence using these n. How do you do that? Well, it is fairly straightforward to do it. At this stage I can just leave it to you, but let us indulge ourselves and actually try to write down a proof . Now, what you do is the following.

Choose first consider $\{n \in \mathbb{N} : |a_n - a| \geq \epsilon\}$. Let us call this set S. What we know is this is an infinite set that is, what I have asked you to prove .

Given that it is an infinite set what you do is . Set $n_1 := \inf S$. Then set $n_2 := \inf(S \setminus s_1$. And I believe you know what is going to come. Set $n_k = \inf(S \setminus \{s_1, s_2, ..., s_{k-1}\})$, and this will give you the required subsequence . So, we have found a subsequence that cannot converge to 'a'. That is, how this subsequence was constructed, that cannot possibly converge to 'a'.

This is a contradiction, so that means, our original hypothesis that a_n does not converge to a is wrong, and this concludes the proof.

This is a course on Real Analysis, and you have just watched the module on subsequences.