Real Analysis - I

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Lecture – 8.3
Limit Laws

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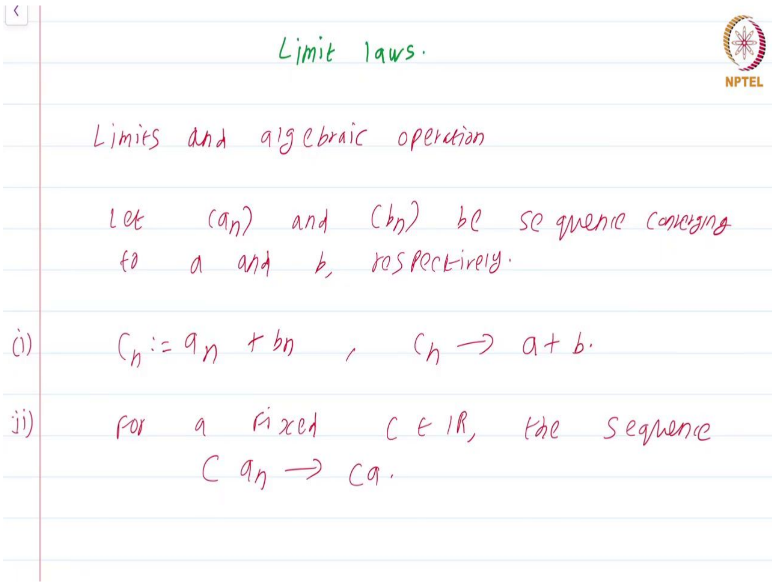

The definition of a limit is quite involved and it is very difficult to show directly from the definition that a particular sequence converges. One drawback of the definition is to even apply the definition, you need to have a candidate for the limit; and then you can use the definition to show that the particular sequence converges to that limit. However, the purpose of the definition is not exactly to use the definition directly to show that limits exists; but rather you prove useful theorems that allow us to manipulate sequences to show that particular sequences have limits and compute those limits also.

So, these are called limits, Limit Laws. So, let me first describe some algebraic limit laws; limit laws, limits and algebraic operations. So, the hypothesis is as follows;

Let a_n and b_n be sequences converging to a and b, respectively.

Then you have the first limit law, the sequence $(a_n + b_n)$; that means, I am defining a new sequence c_n, whose nth term is nothing, but the sum of the sequences a_n and b_n, the nth terms of the sequence a_n and b_n. Then c_n converges to as you can guess a + b.

And the second limit law is for a fixed $C \in \mathbb{R}$, the sequence $(Ca_n) \to Ca$. So, I am no longer using an elaborate notation, I am just denoting the sequence by an expression for its nth term. So the sequence Ca_n converges to Ca.

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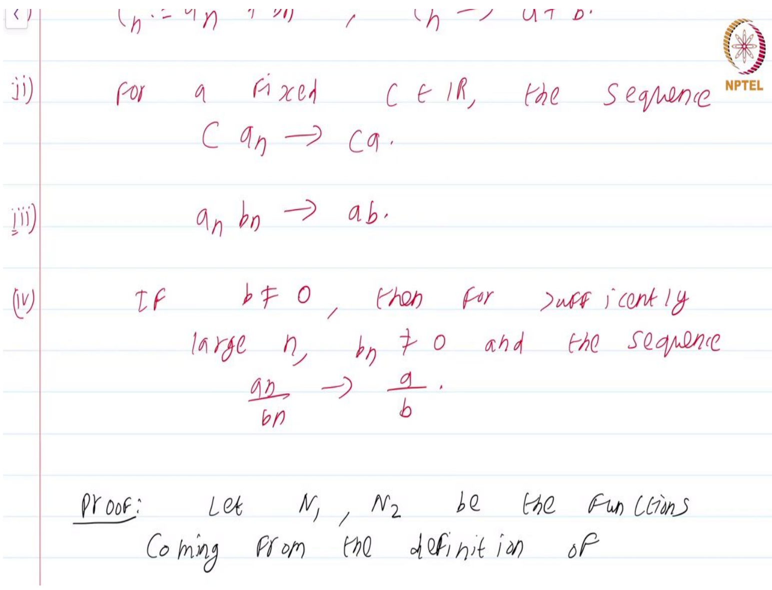

Three, $a_n b_n \to ab$; the product of the sequences converges to the product of the limit.

Fourth point, if $b \neq 0$ then for sufficiently large n, $b_n \neq 0$ and the sequence $\frac{a_n}{b_n} \to \frac{a}{b}$.

Now, here is where I am being a bit imprecise, recall the limit definition when we had discussed the limit definition way back in an earlier module. I had mentioned that many times the sequences will not be defined from the first term, they will be defined after some particular point or sometimes even the index will be negative also.

So, here is a case where this $\frac{a_n}{b_n}$ may not be defined for all n; but the assertion is that, for suitably large n $b_n \neq 0$. So $\frac{a_n}{b_n}$ makes sense and $\frac{a_n}{b_n} \to \frac{a}{b}$. Now, let us proceed with the proof one by one, Proof. So, for the first part I have to show that $a_n + b_n$ converges to a + b.

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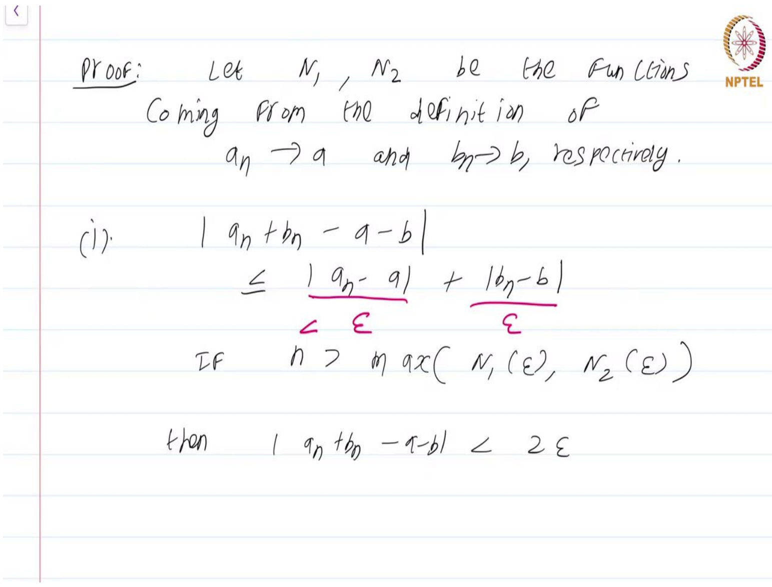

Let N_1, N_2 be the functions coming from the definition of $a_n \to a$ and $b_n \to b$, respectively. That means, these functions N_1 and N_2 will tell you how far along the sequence you have to go to make $|a_n - a|$ and $|b_n - b|$ less than ϵ. Now, observe for the first part, observe that $|a_n + b_n - (a + b)| < |a_n - a| + |b_n - b|$. This is just an application of the triangle inequality and as I remarked earlier; applications of basic inequalities will be done without specific mention as to which inequality I am using.

Especially the triangle inequality and the reverse triangle inequality, I will not hesitate to use it at all; I will use it left and right without even mentioning what I am doing. So, you have $|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b|$. But if $n > \max\{N_1(\epsilon), N_2(\epsilon)\}$, then each one of these terms will be less than ϵ.

Because if n is greater than $\max\{N_1(\epsilon), N_2(\epsilon)\}$, then it is certainly going to be greater than $N_1(\epsilon)$. Therefore $|a_n - a| < \epsilon$. Similarly for the term $|b_n - b|$. So, if $n > \max\{N_1(\epsilon), N_2(\epsilon)\}$, then $|a_n + b_n - a - b| < 2\epsilon$.

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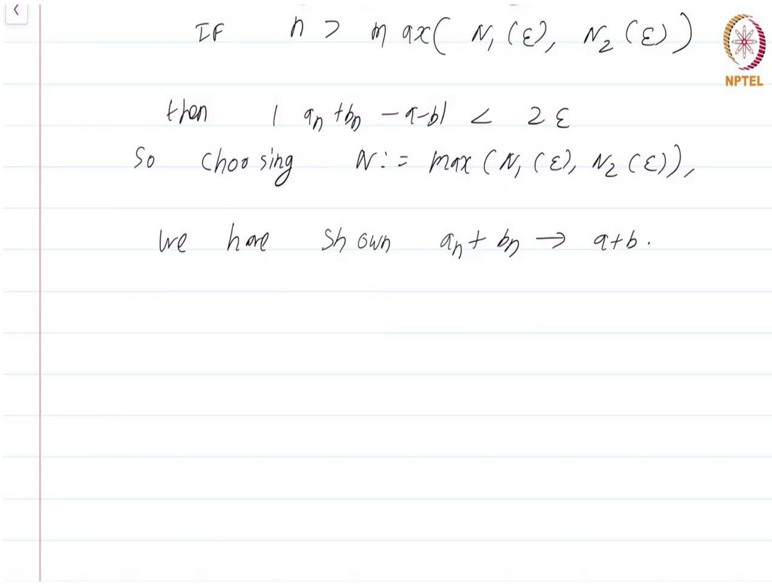

So, choosing the function N to be $\max\{N_1(\epsilon), N_2(\epsilon)\}$; we have shown $a_n + b_n \to a + b$. Note I am implicitly also using $K - \epsilon$, that we talked about in an earlier module.

So, this proof was rather easy, we are just going to choose N function to be $\max\{N_1, N_2\}$.

Now, Part - 2 I am not even going to prove because Part - 2 is a special case of Part - 3.

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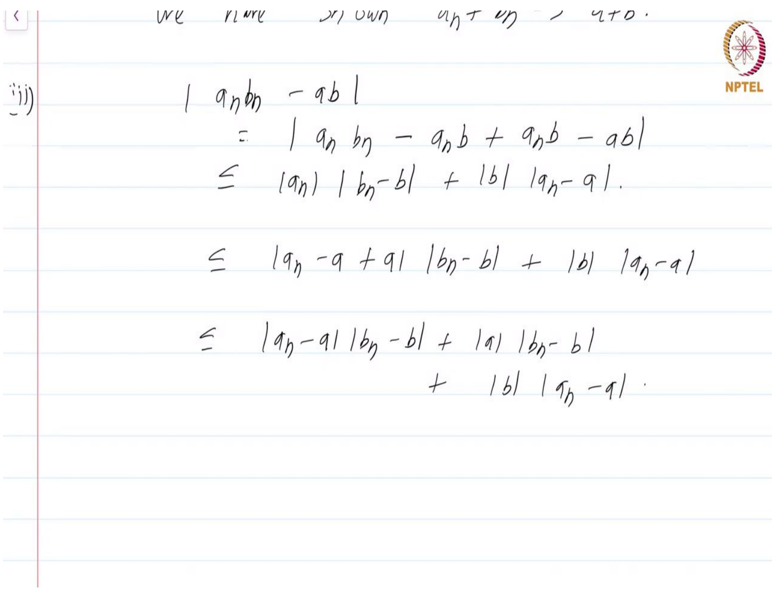

So, let me just directly jump to part 3. Now, what do we have to show? We have to show that $|a_n b_n - ab|$ can be made less than ϵ. Well this is a standard trick we are going to do; the terms that we have control over are terms of the type $|a_n - a|$ and $|b_n - b|$, unfortunately they are missing. But we have become experts at this from high school training; in fact our muscles for this particular thing has been greatly developed by repeated use, what we do is add and subtract an appropriate term.

So, this is same as $a_n b_n - a_n b + a_n b - ab$. I have just added and subtracted the term $a_n b$ from this, which is less than or equal to $|a_n||b_n - b| + |b||a_n - a|$. Now, this in turn is again less than or equal to $|a_n - a + a||b_n - b| + |b||a_n - a|$. I have just now added and subtracted 'a' in the very first term. Now, this is again less than or equal to $|a_n - a||b_n - b| + |a||b_n - b| + |b||a_n - a|$.

Now, again if $n > N(\epsilon)$, where N comes from part 1, $\max\{N_1, N_2\}$.

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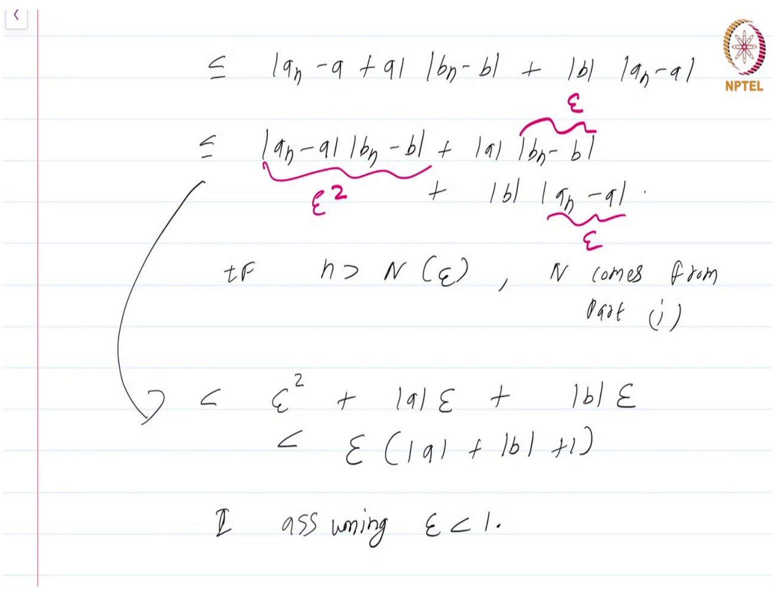

If $n > N(\epsilon)$, what do we have? This very first term is less than ϵ^2; this term is less than ϵ. So, what we get is, this whole thing is less than $\epsilon^2 + |a|\epsilon + |b|\epsilon$, which is less than $\epsilon(|a| + |b| + 1)$.

Here I am assuming $\epsilon < 1$, which I can , I can just choose. If I can show that the quantity that I am interested in $|a_n b_n - ab| < \epsilon$ whenever $\epsilon < 1$ for n sufficiently large; then by the discussions that we had in quite detail earlier, I would get a global function and defined from $\mathbb{R}^+ \longrightarrow \mathbb{N}$ that does the job in the definition of convergence.

So, I have got finally, that $|a_n b_n - ab| < \epsilon(|a| + |b| + 1)$.

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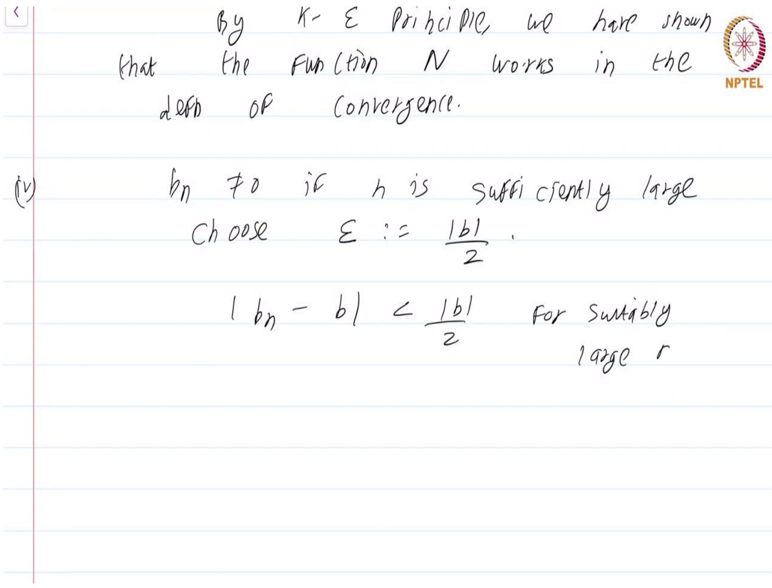

So, again by $K - \epsilon$ principle, we have shown that the function N works in the definition of convergence. So, this concludes the proof of third part. And second part just follows immediately from the proof of this part because the sequence (a_n) is just the constant sequence c. Now, for the final part we have to first show that; or just $b_n \neq 0$, if n is sufficiently large.

Now, , choose ϵ to be defined to be $\frac{|b|}{2}$. Then we know that, $|b_n - b| < \frac{|b|}{2}$ for suitably large n.

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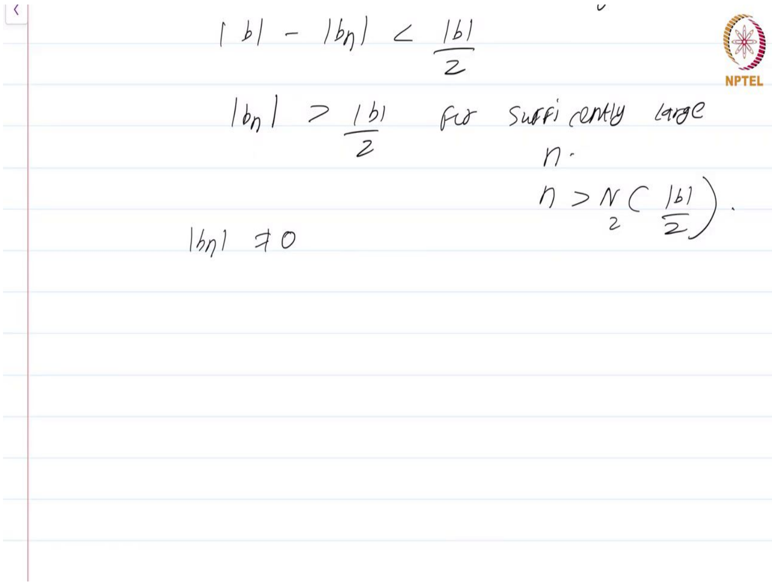

Then which means $|b| - |b_n| < \frac{|b|}{2}$; figure out which inequality from lectures earlier I am using. Now, this shows that $|b_n| > \frac{|b|}{2}$ for sufficiently large n. In fact, we know what this n is going to be; this is just $n > N_2(\frac{|b|}{2})$.

So that means $b_n \neq 0$ for sufficiently large n; simply because $|b| \neq 0$. Now, once we have done this, we are going to show that $|\frac{a_n}{b_n} - \frac{a}{b}|$; I mean it can be made less than ϵ, but instead of doing that what I will do is, I will show, we will show $\frac{1}{b_n} \to \frac{1}{b}$.

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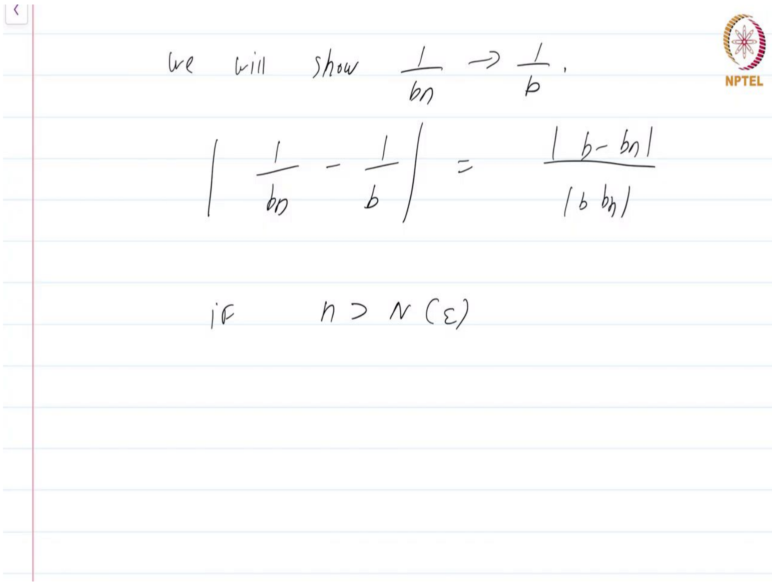

If I can do this, then I can apply part 3 to conclude that $\frac{a_n}{b_n} \to \frac{a}{b}$. So, how will I show that $\frac{1}{b_n} \to \frac{1}{b}$? Well it is just brute force; I look at $\frac{1}{b_n} - \frac{1}{b}$, and this is as we know it is just $\frac{1}{b_n} - \frac{1}{b}$.

Now, again if $n > N(\epsilon)$. Now ϵ is no longer what we define $\frac{|b|}{2}$ or whatever ϵ is now any quantity really doesn't matter in $\mathbb{R}^+$. Then what we get is $|b - b_n| < \epsilon$, so whereas the denominator $|b_n b|$.

Just one moment, let me make one slight change; what I will do is, I will not consider the function N that we had done before. What we will do is, we will be slightly more clever.

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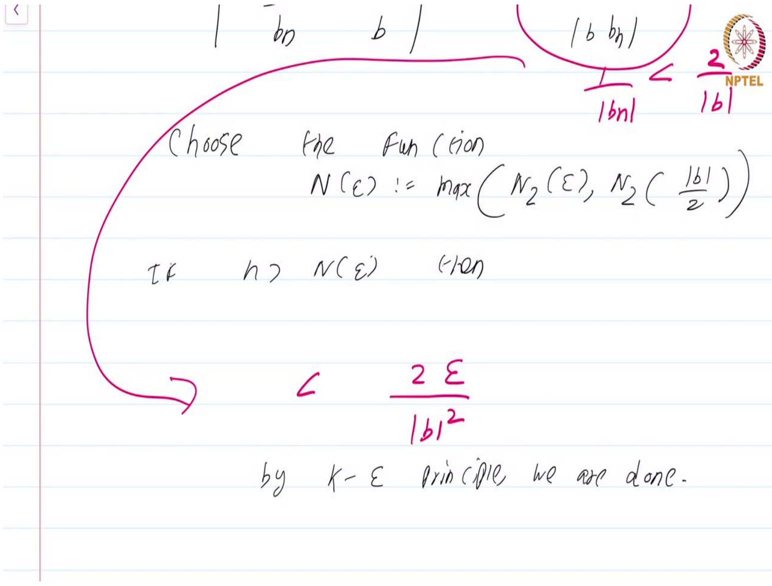

Choose the new function N; remember the N that was there earlier involve the sequence a_n. So, I should not reuse that right now, because it makes no sense; because what I am trying to prove has no involvement of a_n at all.

So, that was a slight goof up, sorry about that. Well choose $N(\epsilon) = \max\{N_2(\epsilon), N_2(\frac{|b|}{2})\}$; you will understand in a moment why I am doing this. Now, what will happen to the numerator if $n > \epsilon$? If $n > N(\epsilon)$, then the numerator $|b - b_n|$ this is certainly going to be less than ϵ. Whereas, the denominator is going to be |b| and is certainly going to be greater than $\frac{|b|}{2}$. $\frac{1}{|b_n|}$ will be less than $\frac{2}{|b|}$.

Why is this the case? Well because by our choice of $N(\epsilon)$ being $\max\{N_2(\epsilon), N_2(\frac{|b|}{2})$. We have already seen that $|b_n|$ will have to be greater than $\frac{|b|}{2}$. If $|b_n| > \frac{b}{2}$, $\frac{1}{|b_n|}$ has to be less than $\frac{2}{|b|}$; which means this whole quantity will be less than $\frac{2\epsilon}{|b|^2}$. I hope you followed this, this is slightly tricky; but it is not that difficult, you have shown that $|\frac{1}{b_n} - \frac{1}{b}| < \frac{2\epsilon}{|b|^2}$. So, again by $K - \epsilon$ principle, we are done.

So, this concludes the various algebraic properties that limits enjoy; always use these algebraic properties to conclude that limits exists, we will see several examples soon enough. Do not try to use the definition except as a last resort. Now, let me end with some more basic properties of limits; this has got to do with limits and order.

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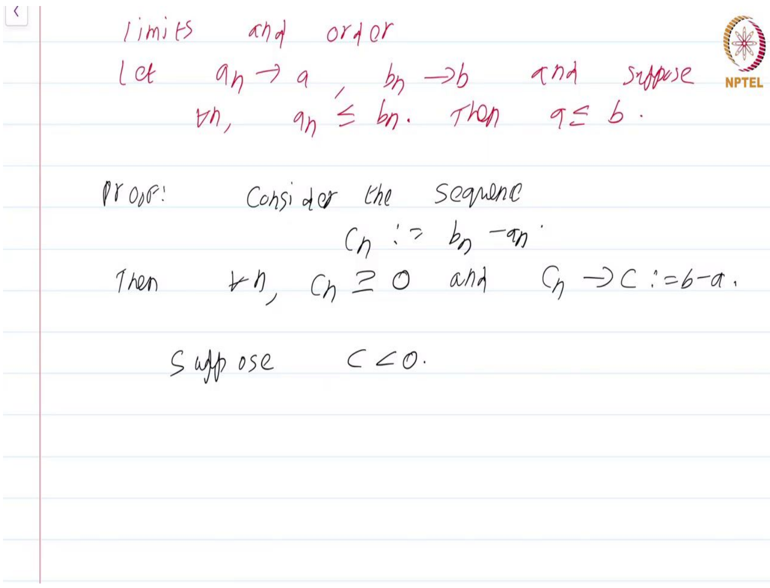

Let again $a_n \to a,\ b_n \to b$ and suppose for all n, you have $a_n \le b_n$, then $a \le b$.

Now, the proof of this is rather easy. Consider the sequence $c_n := b_n - a_n$ then for all n, $c_n \geq 0$. And $c_n \to c$ which is going to be by definition b - a. Why is that? Because the sequence $-a_n \to -a$, and the $(b_n - a_n) \to b - a$.

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Suppose $c < 0$. Then

$$|c - c_n| = |c| + |c_n|$$

$c < 0$ and $c_n > 0$

$$|(-1)(c_n - c)| = |c_n - c|$$

If $\varepsilon := |c|$, then

$$|c_n - c| \geq \varepsilon$$

This means $c_n \to c$ is impossible. This contradiction finishes the proof.

Now, suppose c < 0, we will arrive at a contradiction. Then $|c - c_n| = |c| + |c_n|$. Why is this the case? Well observe that mod modulus c is less than 0; observe that c is less than 0, and c n is greater than 0, right.

So, how did I get from this that $|c - c_n| = |c| + |c_n|$. Well, because c < 0 and $-c_n < 0$. So this quantity I could have written it as $|(-1)(c_n - c)|$, which is same as $|c_n - c|$. I am just reversing the thing in an extremely unnecessarily complicated way.

And c_n is a positive quantity and -c is also a positive quantity. So the modulus of this is just $|c| + |c_n|$, just from the definition of modulus or absolute value. That means if I choose $\epsilon = |c|$, then $|c_n - c|$ is always greater than ϵ. Because $|c_n|$ is, rather it will be greater than or equal to $|c_n - c|$; will always be greater than or equal to ϵ. This means $c_n \to c$ is impossible. So, this contradiction, finishes the proof.

So, one last theorem involving limits and order, this is called the squeezing theorem, squeezing or sandwich theorem.

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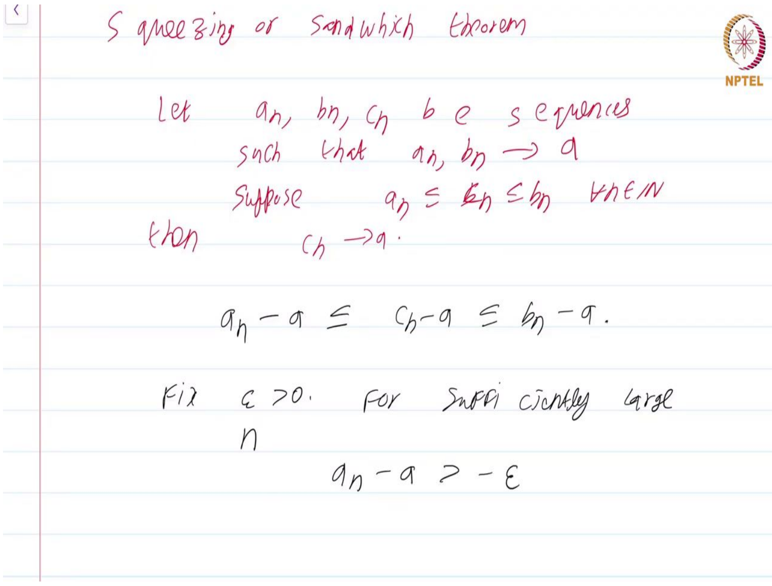

Let a_n, b_n, c_n be sequences such that a_n and b_n both converge to the same quantity 'a'. Suppose c_n is squeezed in between a_n; $a_n \leq c_n \leq b_n \ \forall n \in \mathbb{N}$, then $c_n \to a$.

Again the proof of this is fairly straightforward. Observe that $a_n - a \leq c_n - a \leq b_n - a$ for all n. All I have done is, I have taken the inequality $a_n \leq c_n \leq b_n$ and subtracted a from all the sides.

Now, fix $\epsilon > 0$ for sufficiently large n. We simultaneously can guarantee that $a_n - a > -\epsilon$ and $b_n - b < \epsilon$. Why? I want you to think about why we can do this.

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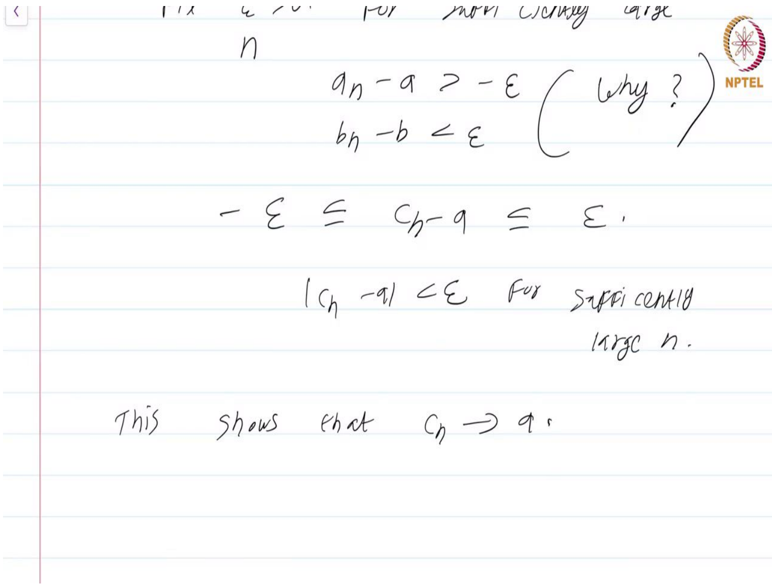

So that means we have $-\epsilon \leq c_n - a \leq \epsilon$. In fact, you will have strict inequalities, but less sometimes just for convenience. I will not keep track of whether something is less than or less than or equal to. Any statement where I replace a less than by less than or equal to is still true, though not optimal. That is not really relevant to the proof.

So that means, $|c_n - a| < \epsilon$ for sufficiently large n. This shows that $c_n \to a$. So, this concludes this proof.

This is a course on Real Analysis and you have just watched the module entitled Limit Laws.