Real Analysis - I
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Lecture – 8.2
A Descriptive Language for Convergence

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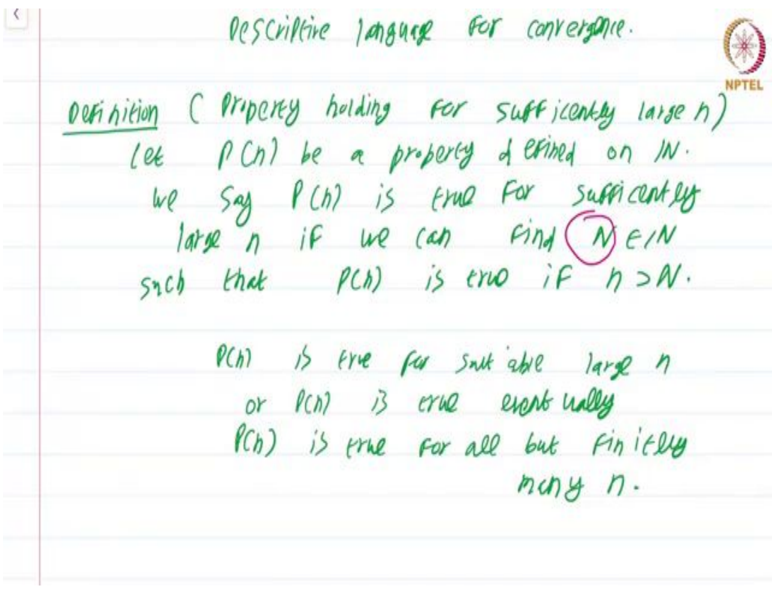

The various definitions of Convergence that we have given are all quite involved. Now that we have made the effort to master these definitions, there is no necessity to be extremely verbose when writing down proofs or studying convergence or divergence of various sequences. It is very useful both from the perspective of elegance as well as clarity to use language in our aid to communicate in an effective manner.

So, let me introduce a few definitions that will make it easy to talk about convergence and divergence in a precise manner, but at the same time not too wordy. Definition, this is the definition of property holding for sufficiently large n. So, this is going to define what it means for something to be true for sufficiently large n.

Let $P(n)$ be a property defined on the natural numbers. We say $P(n)$ is true for sufficiently large n, if we can find, we can find $N \in \mathbb{N}$ such that $P(n)$ is true if $n > N$, ok. So, what this definition is saying is that the property P should be true for sufficiently large n means after some natural number for sufficiently large natural numbers or for suitably large natural numbers $P(n)$ is true and that is captured by saying that there is some point N beyond which this is true.

Now, notice this funny thing that property holding for sufficiently large n does not specify how large we need to go. It could be the case that we may have to go n greater than 10 million or 10 trillion or 10 trillion million. It could be very very large that does not matter, ok. So, we will usually say that $P(n)$ is true for sufficiently large n.

We will also use certain synonyms just for variety to spice up life, will say $P(n)$ is true for suitably large n; suitably large n or $P(n)$ is true eventually; is true eventually or even a fancier one which is also the same but it is useful in many other contexts as well. $P(n)$ is true for all but finitely many n. These are various versions, various phrases that we shall use to describe the same thing.

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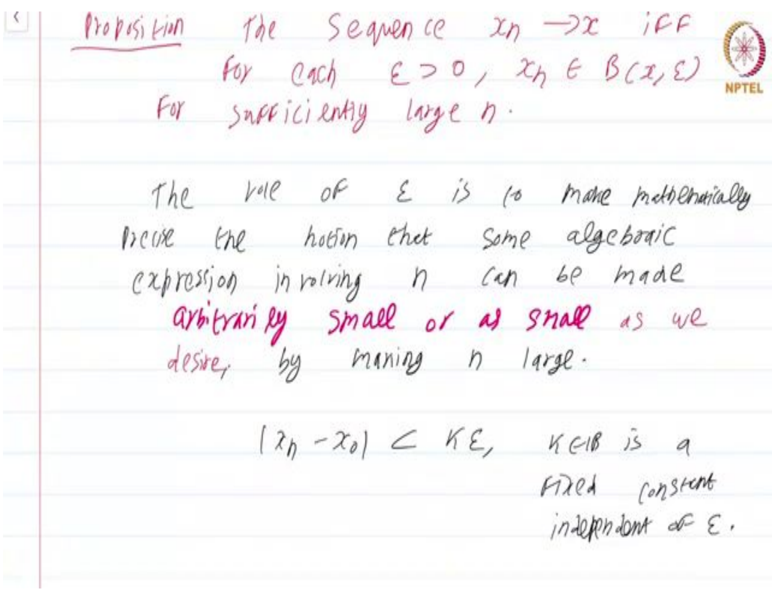

So, now that we have this additional terminology, the definition of convergence becomes rather easy which I state as a proposition and I leave it to you as an exercise because it is just a matter of unwinding the definitions. The sequence x_n converges to x if and only if for each $\varepsilon > 0$, $x_n \in B(x, \varepsilon)$ for sufficiently large n; for sufficiently large n. This is just a reformulation and as you can see we have gained a great economy in our expression, ok.

So, now let me make some more remarks. Note the role of ε in each of the definitions that I have stated, ok, there are several I think 4 by my count. The role of ε is that it is a fixed quantity, but otherwise arbitrary and our goal is to show usually when you are showing convergence, the goal is to show that some particular algebraic expression can be made less than this\varepsilon. This algebraic expression is $|x_n - x_0|$. We have to somehow make it less than ε. Of course, we have flexibility in the small n, we can choose this small n by our desire to make it really large but at the end of the day we have to show that this expression can be made less than ε.

In other words, the role of ε is to make precise, make mathematically precise the notion that a particular algebraic expression that involves a variable n can be made arbitrarily small or as small as we desire by increasing n. So, the role of ε; the role of ε is to make mathematically precise the notion that some expression; that some algebraic expression involving n can be made; can be made arbitrarily small. That is exactly what or as small as we desire that is the role of ε.

Let me be precise as small as we desire by making n large, right. And how do we achieve this? You will see in the module on examples that we achieve it by manipulating the expression as much as possible. Sometimes it is easy algebraic manipulations, sometimes it is messy algebraic manipulations and often what will happen is once you do all the manipulations to show that $|x_n - x_0|$ is less than ε, you end up with something like $|x_n - x_0| < K\varepsilon$ where $K \in \mathbb{R}$ is a fixed constant, the fixed constant independent of ε; independent of ε.

In other words, instead of getting $|x_n - x_0| < \varepsilon$, we may end up with getting $|x_n - x_0| < 3\varepsilon$. Now, of course if you have been following what I am saying this should not really matter because the role of ε is to say that $|x_n - x_0|$ can be made arbitrarily small. If it can be made 3 times arbitrarily small; 3 times arbitrarily small is still arbitrarily small, this is of no consequence and we illustrate this with the following lemma which makes precise what I am trying to say.

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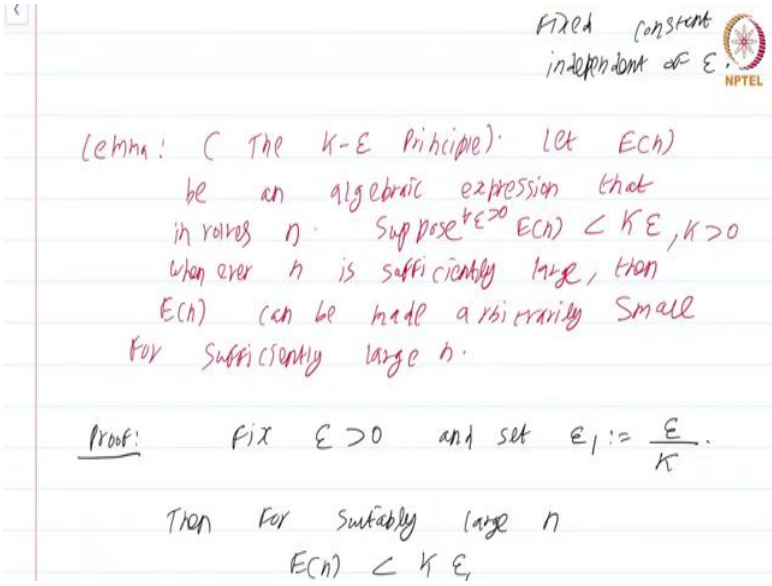

Lemma: this is called the $K - \varepsilon$ principle. Let $E(n)$ be an algebraic expression that involves n. Suppose, we are not very satisfied with this particular phrase let $E(n)$ be an algebraic expression that involves n, you can just say E is a function from the natural numbers to the real numbers, ok.

Suppose $E(n) < K\varepsilon$ whenever n is sufficiently large. So, this is an opportunity for you to see our new enhanced vocabulary in action. Then $E(n)$ can be made arbitrarily small for sufficiently large n. What do I mean by $E(n)$ can be made arbitrarily small for sufficiently large n that you will see in the proof, but you should have guessed what this means from the discussion that we had about. Proof; what we have is that $E(n)$ can be made less than $K\varepsilon$ whenever n is sufficiently large. So, fix $\varepsilon > 0$ and set $\varepsilon 1$ to be.

Just one moment, there is one catch that I forgot to mention. Suppose $E(n) < K\varepsilon$, where $K > 0$, this is crucial ok. Fix $\varepsilon > 0$ and set $\varepsilon_1 = \frac{\varepsilon}{K}$.

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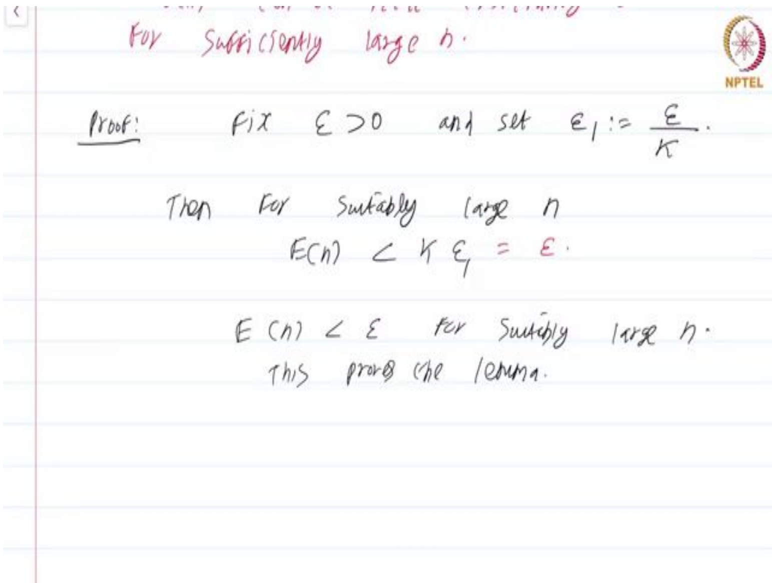

Then for suitably large n; suitably large n, $E(n) < K\varepsilon_1$. Why is this the case? Well, In the hypothesis that $E(n) < K\varepsilon$ whenever n is sufficiently large that ε could be any number whatsoever, ok. What we have done is, we have chosen that ε to be ε_1, so to make sure that there is no confusion whatsoever.

Suppose for each $\varepsilon > 0$, $E(n) < K\varepsilon$, $K > 0$ whenever n is sufficiently large. Suppose you can do that for each\varepsilon in particular, you can do it for the choice of ε to be ε_1, ok. So, $E(n)$ can be made less than $K\varepsilon_1$ whenever n is suitably large, but this is just ε right; this is just ε. What this is saying in the end is that $E(n)$ can be made less than ε for suitably large n; for suitably large n.

This just says that $E(n)$ can be made arbitrarily small. The definition of can be made arbitrarily small is as you increase n, it is possible to make $E(n)$ less than ε. So, this proves the lemma; this proves the lemma. So, this lemma, I will not explicitly invoke in the proofs. If it gets too messy in the proofs in some proof of convergence or I will just leave the expression on the right hand side as some constant time ε and just say that we are done. I will not explicitly invoke this particular lemma. This is a course on Real Analysis and you have just watched the module entitled "A Descriptive Language for Convergence".