Real Analysis - I
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Lecture – 8.1
Deep Dive into the Definition of Convergence

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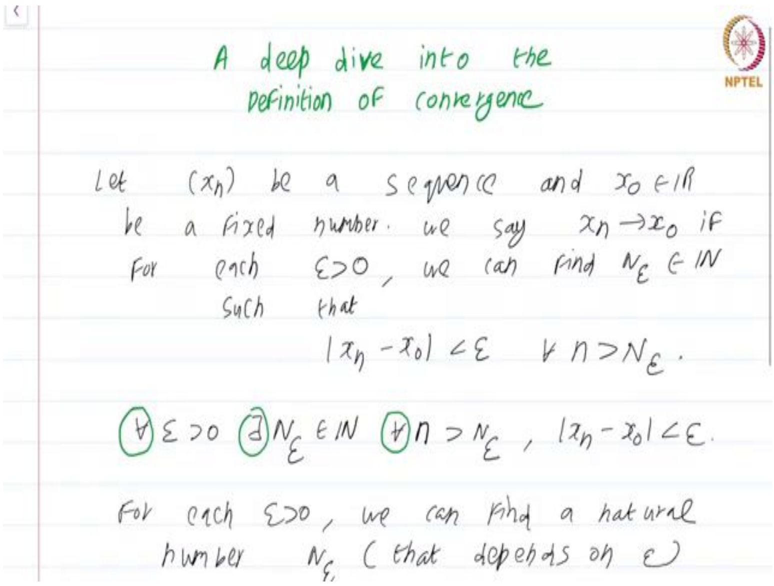

In this module we are going to take a Deep Dive into the Definition of Convergence. Let me recall the definition as it is presented in most textbooks; let x_n be a sequence and $x_0 \in \mathbb{R}$ be a fixed number. We say x_n converges to x_0 if for each $\varepsilon > 0$; we can find $N_\varepsilon \in \mathbb{N}$ such that $|x_n - x_0| < \varepsilon$ for all $n > N_\varepsilon$.

Now the order of the quantifiers in this definition is extremely important. To see this let us first rewrite this definition in logical symbolism.

$\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N}$, such that $\forall n > N_\varepsilon, |x_n - x_0| < \varepsilon$.

So, it is at this point that it might be a good idea to read this symbol as for each; for each $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$, such that for all $n > N_\varepsilon, |x_n - x_0| < \varepsilon$. It does not matter how you read the second one, even if you read it as for all or for each, you will not, you are less likely to be misled; but reading the first one as for all can lead to some issues, unless you precisely understand what for all means, ok.

Now, it is an interesting exercise now to translate this translation back into the original language that of English. What is this saying? Well for each $\varepsilon > 0$, we can find a natural number N_ε that depends on ε, that is crucial, that depends on ε, such that whenever $n > N_\varepsilon$, we have $|x_n - x_0| < \varepsilon$.

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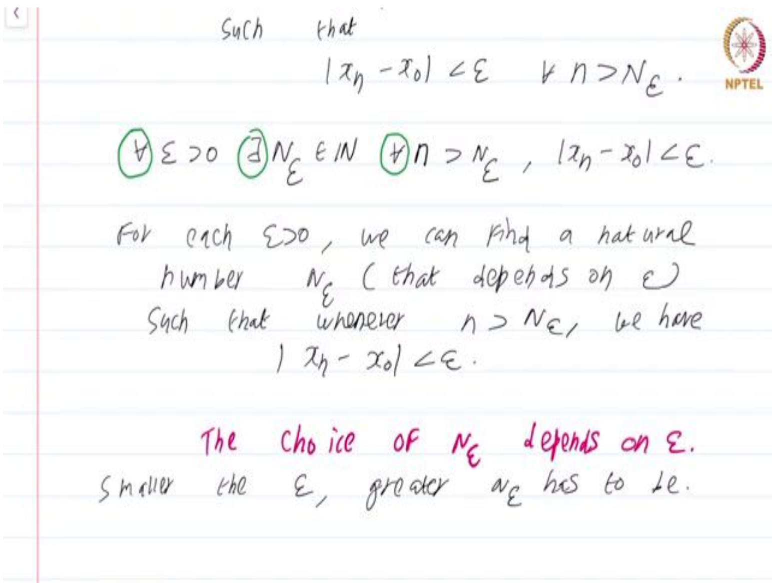

So, we started off with the definition which was more or less in English; we translated it to logical symbolism, now we have translated back. So, the key point that I want you to take from this is that, the choice, the choice of N_ε depends on ε, depends on ε.

So, in general, smaller the ε, smaller the ε, greater N_ε has to be, this is just a general statement; it is not always the case, but in general you have to choose higher N_ε whenever ε is smaller. Essentially if ε is really small, we have to go further along the sequence, so that the terms of the sequence are at max ε distance away from the limit, ok.

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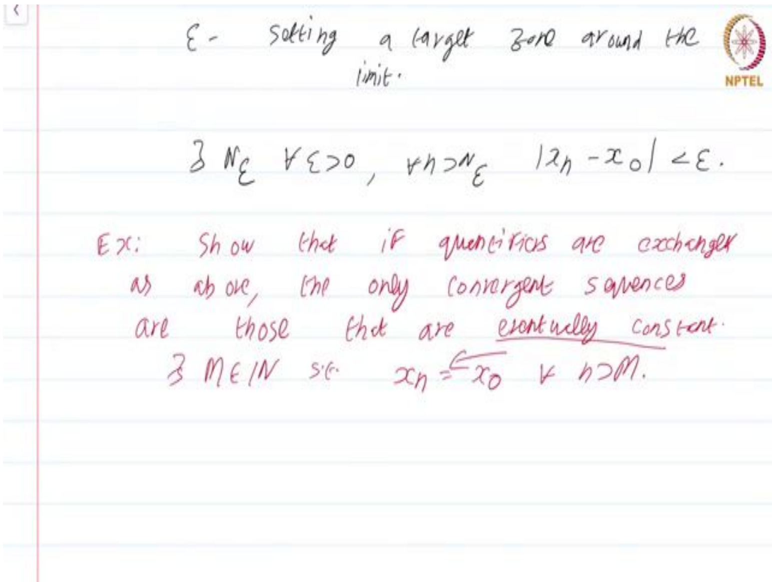

So, you should think of ε as setting a target zone; as setting a target zone around the limit; around the limit; and the role of N_ε is to ensure that x_n, whenever $n > N_\varepsilon$ is always within the target zone,.

Now, the reason why we have chosen to present our definition of limit with a function capital N that depends on ε is to highlight the fact that N_ε crucially depends on ε. Furthermore, several proofs will become easier, if you treat capital N_ε as a function of ε, rather than just a quantity that depends on it. To make the relationship explicit using a function makes our life easier; we will see that when we study many examples of convergent sequences and showing that sequences converge to a particular limit.

Now, if we had interchanged the role of ε and N_ε, that is interchange the quantifiers, we will get the following statement; $\exists N_\varepsilon$ such that $\forall \varepsilon > 0, \forall n > N_\varepsilon, |x_n - x_0| < \varepsilon$. Suppose I interchange for each $\varepsilon > 0$ and there exists N_ε, writing there exists N_ε for all $\varepsilon > 0$. So, we have the same definition; things go badly wrong, and this I leave it as an exercise for you.

Exercise: Show that if quantifiers are exchanged as above, the only convergent sequences; the only convergent sequences are those that are eventually constant. What do I mean by eventually constant? Well that means, eventually constant just means; $\exists M \in \mathbb{N}$ such that $x_n = x_0$ for all $n > M$. After a particular point the sequence just becomes a constant sequence, where that constant is nothing, but the limit x_0, ok. Please work out this exercise.

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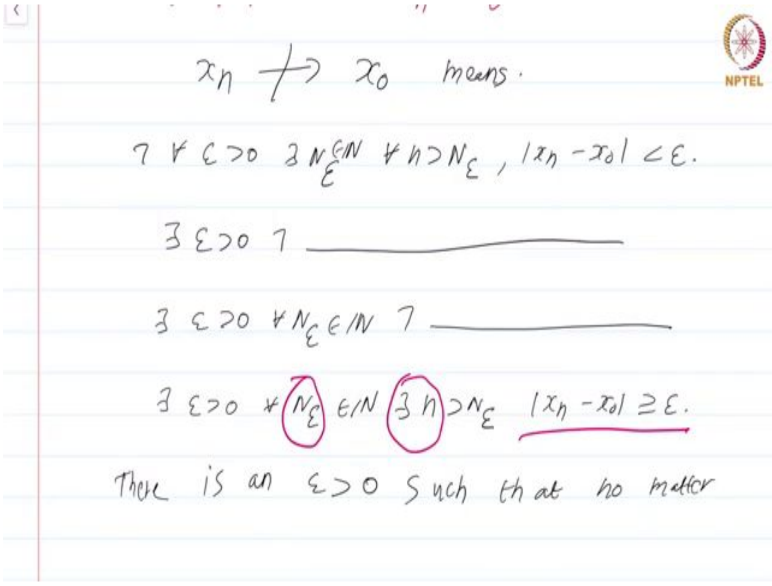

Now, what we are going to do now is, we have seen what convergence, x_n converges to x_0 means in great detail. Even go further in our understanding; we have to first understand what x_n does not converge to x_0 means, what does this mean? We have already grappled with this once when we saw that limits are unique; we had to see what it means for x_n does not converge to x_0.

The best way to do this is to actually just formally negate the definition in logical symbols. What was the definition in logical symbols? It was just

$\forall\varepsilon > 0 \exists N_\varepsilon \in \mathbb{N}$, such that $\forall n > N_\varepsilon, |x_n - x_0| < \varepsilon$.

This is the definition of x_n converges to x_0.

Well we have to negate this; the best way to do it is to keep pushing the negation inside, that is the way to do it. Now, when you first push the negation inside, inside the $\forall\varepsilon > 0$, what we get is, $\exists\varepsilon > 0$, then negation of the rest of the statement, right. Now, you push it inside the statement, the part $\exists N_\varepsilon \in \mathbb{N}$; so that will just give you, $\exists\varepsilon > 0 \forall N_\varepsilon \in \mathbb{N}$, then negation of the rest.

Well what is the next step? Well that is fairly easy, we just repeat what we have written so far and we have to change this $\forall n > N_\varepsilon$ to $\exists n > N_\varepsilon$. Then the final negation goes into $|x_n - x_0| < \varepsilon$; the negation of that is $|x_n - x_0| \geq \varepsilon$, ok.

What is this saying? There is an $\varepsilon > 0$, such that for all $N_\varepsilon \in \mathbb{N}$; there exists some $n > N_\varepsilon$, such that $|x_n - x_0| \geq \varepsilon$. What is the saying? For x_n converging to x_0 to fail, for some target ε, it is not the case that every term in the sequence beyond a particular point is fully contained within the target zone.

So, that merely says that, there is some ε, no matter how further along I am on the sequence, that is what the role of N_ε is; no matter where I am in N_ε, there is always some other term greater than N_ε that spoils the day. So, this is what the negation of the definition of x_n converging to x_0 says, ok.

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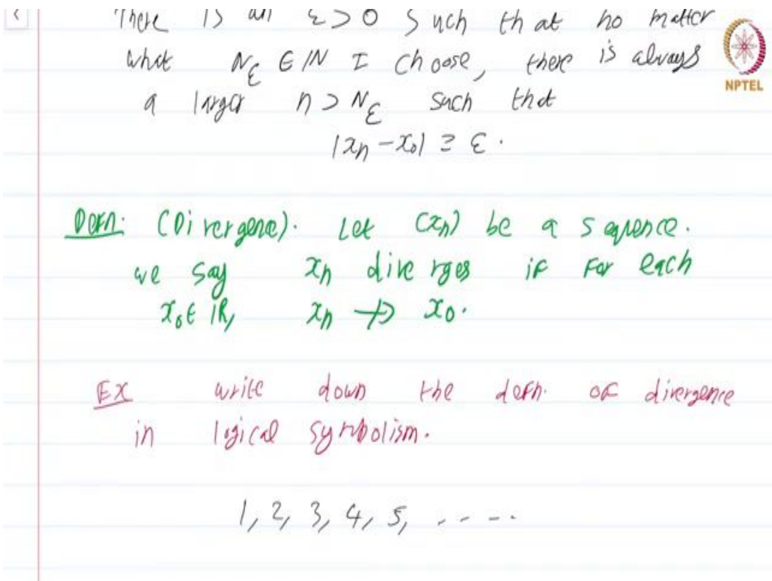

Now, again it is best to translate back into English; there is an $\varepsilon > 0$, such that no matter what $N_\varepsilon \in \mathbb{N}$ I choose, there is always a larger natural number n, such that $|x_n - x_0| \geq \varepsilon$. So, this is the negation of the definition of convergence.

Now, let me give one definition now. Since we saw what is the meaning of x_n does not converge to x_0; the next definition is that of divergence, is that of divergence. Let x_n be a sequence; we say x_n diverges if for each $x_0 \in \mathbb{R}$, x_n does not converge to x_0, ok. No matter what choice of x_0 I choose; it is not the case that x_n converges to x_0. And I immediately give you an exercise, exercise; write down the definition of divergence in logical symbolism.

Now, there are many ways by which a sequence could be divergent; one interesting way is how the sequence 1, 2, 3, 4, 5 and so on which is clearly a divergent sequence. This seems to have a special property that the elements of the sequence get larger and larger; in some sense this converges to $+\infty$, ok. So, it is convenient to have a special terminology for such sequences.

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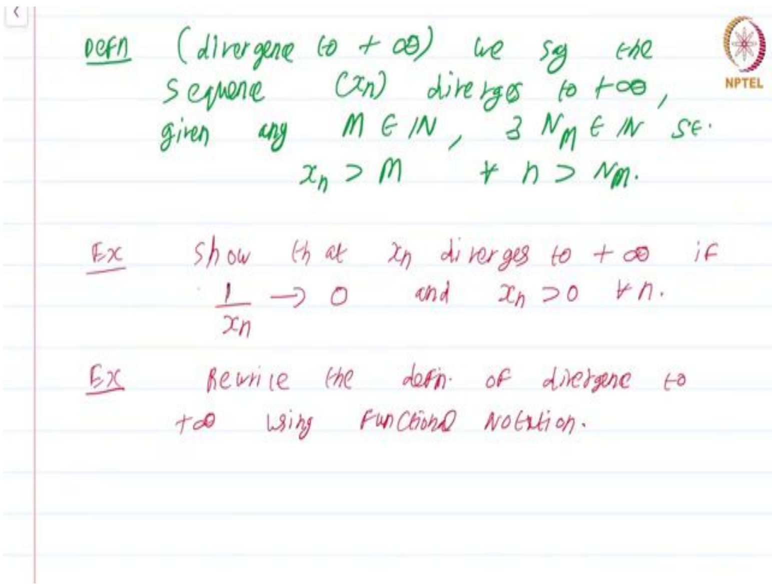

So, I will introduce another definition, Definition: Divergence to$+\infty$, we say the sequence, x_n diverges to $+\infty$, now I am going to write down a statement which is going to be very very similar to the usual definition of convergence; the key thing is we want to capture that the terms of the sequence x_n get larger and larger. So, what we do is, given any $M \in \mathbb{N}$, exists $N_M \in \mathbb{N}$ such that $x_n > m$ for all $n > N_M$, ok. All this is saying is, if you move sufficiently far away through the sequence; then the terms of the sequence will be larger than M and this must be true for each $M \in \mathbb{N}$, ok.

No matter what fixed natural number I take; after some point in the sequence, the terms of the sequence will be greater than M. So, this is fairly straightforward, now that we have spent quite a lot of time digesting the definition of convergence. Again some more exercises for you. Show that, x_n diverges to $+\infty$ if $\frac{1}{x_n}$ converges to 0 and $x_n > 0$ for all n. This is not the best statement that you can make, but this is a straightforward exercise to solve, to show that a particular sequence x_n diverges to $+\infty$, if $x_n > 0$; all you have to do is show $\frac{1}{x_n}$

converges to 0, this is not the absolute best statement that we can make, but this is a good start.

Another exercise for you, rewrite the definition of divergence to $+\infty$ using functionals, functional notation; exactly like what we had for the convergence of sequences, where we used a function $N(\varepsilon)$, rewrite this definition also using functional notation, ok. So, this is about divergence to $+\infty$.

Now, one final version of the definitions above; this definition that I am about to give I will not expand upon it right away, we will see more about this in the chapter on topology.

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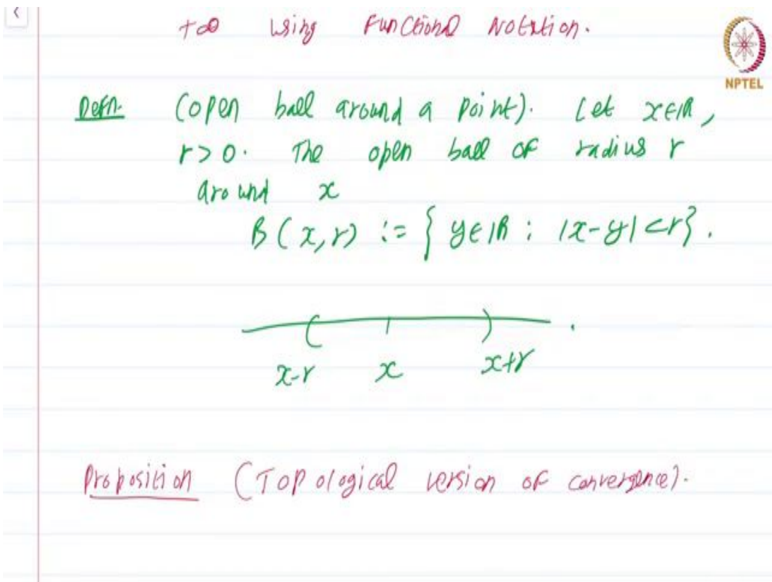

Let me first give a definition, this is the definition of open ball around a point. Let x be a real number and $r > 0$; the open ball of radius r around x, this is denoted $B(x, r)$, this is by definition equal to, $B(x, r) = \{y \in \mathbb{R} : |x - y| < r\}$.

Pictorially if this is the point x, this will just be the open interval $(x - r, x + r)$. Why this is called an open ball, will become very clear to you in the latest sections on the chapter on topology, ok. Now, I will not expand upon this definition right now; but I will leave you with one more definition and a final exercise for this.

Definition, rather let me just phrase it directly as an exercise, proposition, this is called the topological version, topological version of convergence.

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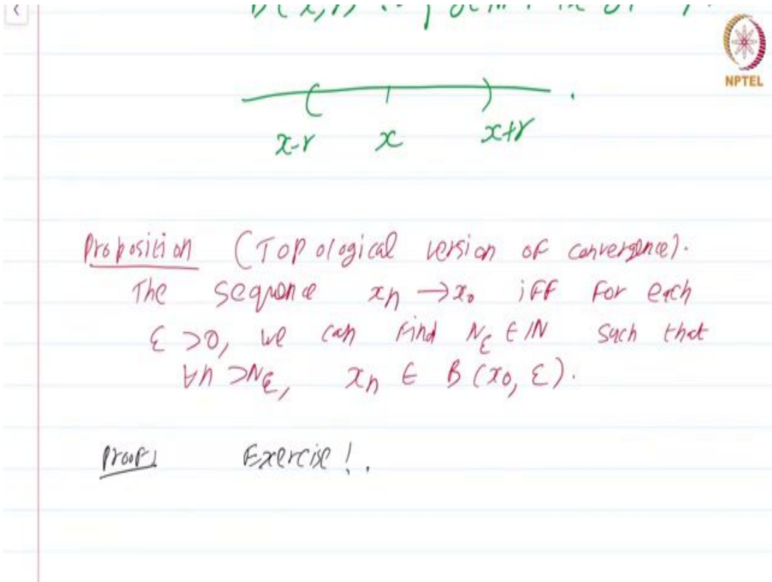

The sequence, x_n converges to x_0 if and only if for each $\varepsilon > 0$; we can find $N_\varepsilon \in \mathbb{N}$ such that for all $n > N_\varepsilon$, $x_n \in B(x_0, \varepsilon)$, the open ball of radius ε centred at x_0. The open ball of radius ε centered at x_0.

Proof, exercise, this is yet another formulation of convergence and this version of convergence, this definition this version of the definition of convergence will prove to be very interesting and important on the tab in the chapter on topology. This is a course on Real Analysis and you have just watched the module titled a Deep Dive into the Definition of Convergence.