Real Analysis - I

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Lecture – 7.4
Uniqueness of Limits

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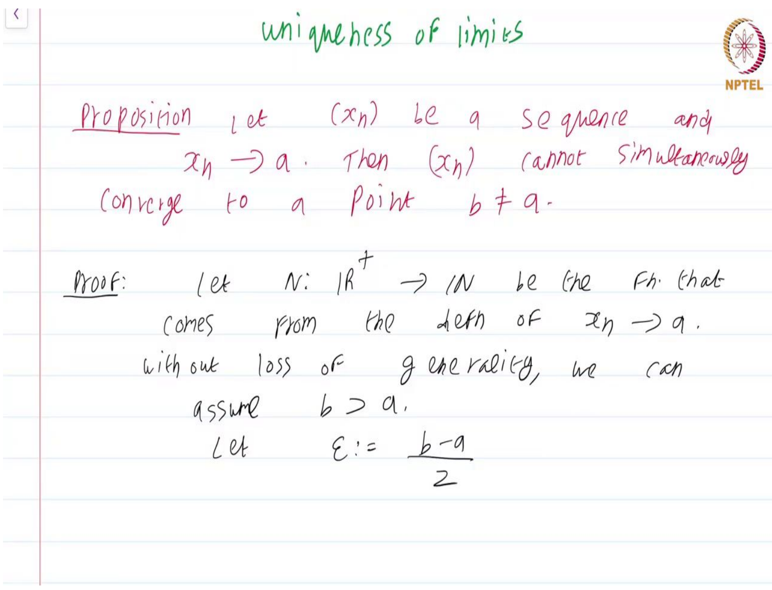

In this short module we will just prove one proposition that is the Uniqueness of limits. Proposition. Let x_n be a sequence sequence and x_n converges to a, then x_n cannot simultaneously converge to a point $b \neq a$. In other words, the limit of a sequence if it exists is unique. Proof. Well, let $N : (0,1) \longrightarrow \mathbb{N}$ be the function that comes from the definition of x_n converging to a, ok. So, instead let me just not take it from $(0,1)$, let me just take it from $\mathbb{R}^+ \longrightarrow \mathbb{N}$. Recall that it does not really matter if I had a function from $(0,1)$ to $\mathbb{N}$ that satisfies the definition of convergence, you get a function from $\mathbb{R}^+$ to $\mathbb{N}$ as well. You will understand in a moment why I had decided to choose the function from $\mathbb{R}^+$ to $\mathbb{N}$.

Now, without loss of generality we can assume that $b > a$, ok. You please go through the entire proof and check really that this assumption of $b > a$ is really actually without loss of generality. Let $\varepsilon = \frac{b-a}{2}$.

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Let $n > N(\varepsilon)$, we must have

$|a-b| \le |x_n - a| + |x_n - b|$.

But $|x_n - a| < \varepsilon$, and

$|a-b| = 2\varepsilon$.

$2\varepsilon \le \varepsilon + |x_n - b|$.

$|x_n - b| \ge \varepsilon \quad \forall n > N(\varepsilon)$.

For our choice ε, $|x_n - b| \ge \varepsilon$ whenever $n > N(\varepsilon)$. This means, it is impossible to satisfy the defn. of convergence for $x_n \to b$.

I am just choosing an $\varepsilon = \frac{b-a}{2}$, let $n > N(\varepsilon)$. Now you should have figure out why I chose $N : \mathbb{R}^+ \longrightarrow \mathbb{N}$, there is no guarantee that $\frac{b-a}{2} < 1$, that is not guaranteed, ok.

Let $n > N(\varepsilon)$. Then we must have; we must have $|a-b| \le |x_n - a + x_n - b|$, right. But $|x_n - a| < \varepsilon$, why is that the case, simply because we choose $n > N(\varepsilon)$, ok. And $|a-b| = 2\varepsilon$ it is just 2ε by the very definition as $b - a = 2\varepsilon$, ok.

What this tells us; what this tells us is that $2\varepsilon \le \varepsilon + |x_n - b|$, right. That means, $|x_n - b| \ge \varepsilon$ and this is true for all $n > N(\varepsilon)$. If you think about this, what this is really saying is that, for our choice of ε; our choice of ε, $|x_n - b| \ge \varepsilon$ whenever $n > N(\varepsilon)$.

This means, it is impossible; it is impossible to satisfy the definition of convergence for x_n converging to b. For this choice of ε, I can never find an $N(\varepsilon)$ such that if $n > N(\varepsilon)$, $|x_n - b| < \varepsilon$, that simply not possible because after a particular stage, more precisely our choice $N(\varepsilon)$, the sequence is always at least ε distance away from the point b.

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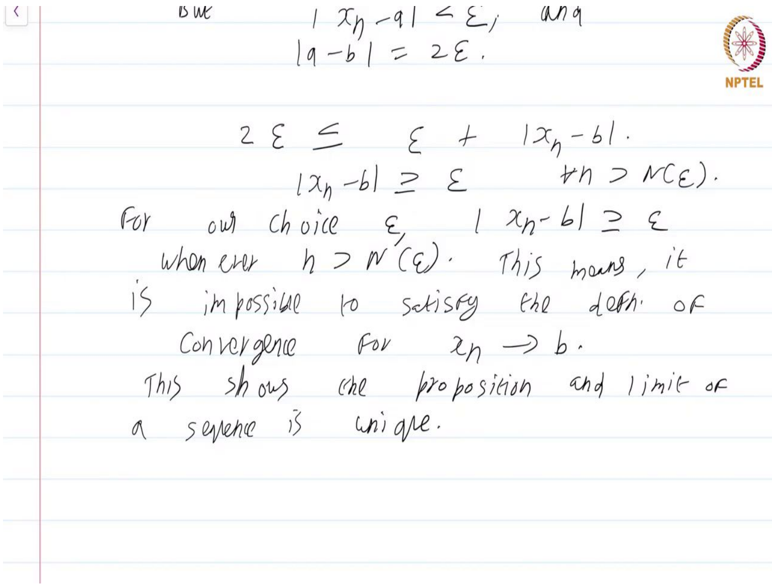

So, the definition can never be satisfied for x_n converging to b. So, this shows the proposition and limit of a sequence is unique. This is a course on Real Analysis and you have just watched the module on Uniqueness of Limits.