Real Analysis - I
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Lecture – 7.2
Definition of Sequence and Examples

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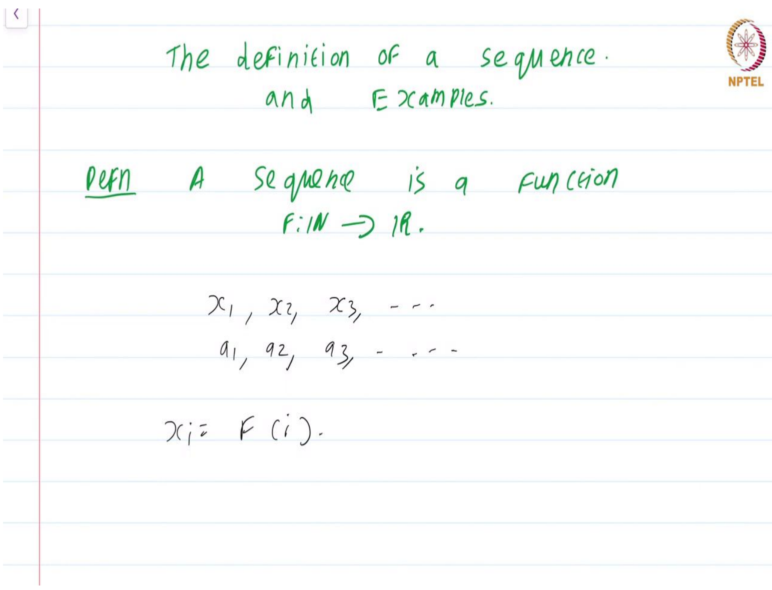

In this module, we will define what a sequence is and give several examples. Definition, a sequence is a function $F : \mathbb{N} \longrightarrow R$, that is it; probably the shortest definition you have seen, even the definition of empty set was slightly bigger I believe. So, we will be studying sequences, this notation, this functional notation is not always convenient; so usually the simplest way to describe a sequence is to just list out the elements as $x_1, x_2, x_3, \cdots$ or $a_1, a_2, a_3, \ldots$.

This is sort of like a dynamic notation for sequences; often we will just list three or four terms and it should be apparent what the rest of the terms are, you should be able to guess what the rest of the terms are, ok. So, this x_i is just supposed to be $F(i)$. So, essentially sequence are just lists, please connect back this to the definitions of countability.

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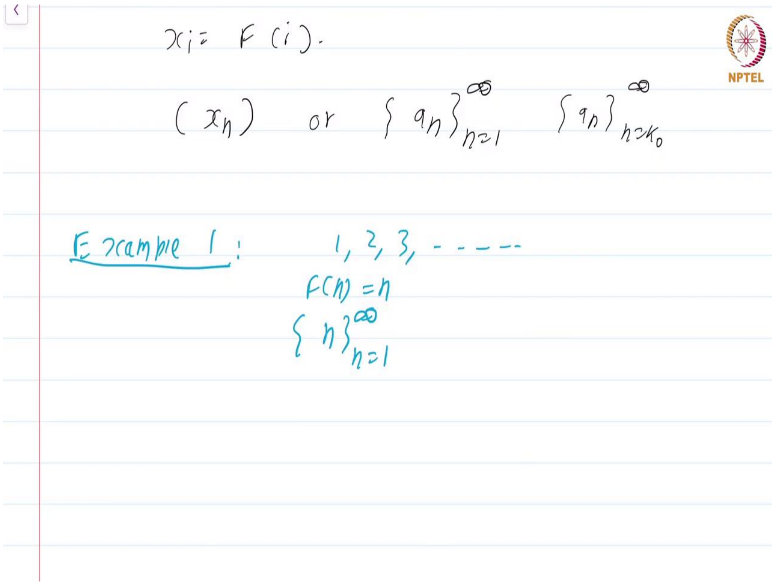

More precise notations for sequences is something like this; you just put x_n in the parentheses, (x_n), or if you want to be ultra-precise, you can use something like $\{a_n\}_{n=1}^{\infty}$.

The last notation has the advantage that I can always change it and say $\{a_n\}_{n=k_0}^{\infty}$, ok. So, we can always start a sequence not from just x_1, but from some further point along; sometimes even negative, you can start from x_{-3}, x_{-2} so on. These are minor modifications to the definition of a sequence; so we will not be ultra precise in defining them, however we might occasionally use such sequences which do not begin at x_1.

So, in the future we will just use any one of these notations for our sequences. Now sequences are very important in analysis, because many of the major concepts of analysis including limits, continuity, differentiation, integration and even sum of infinite series can all be phrased in terms of sequences.

So, our objective is to study, what it means for a sequence to converge to a particular number or approach a particular number or limit of a sequence to be a particular number. Before we get to that, it is always good to see lots of examples.

Usually I recommend we see seven examples at least; simply because seven is the smallest number, but a random person on the street may not be able to identify whether it is prime or not.

Please do not test it in the real world, this is just my personal experience. Let us just see some examples of sequences, example 1; consider the sequence $1, 2, 3, \ldots$. So, if you want to be ultra precise, this is the sequence defined by $F(n) = n$, $n = 1, , \infty$ or I mean I am going to be done here, I am not going to belabour this anymore.

So, this is a sequence of real numbers that is ever increasing, clearly it does not tend to a point on the real line; but in an intuitive sense this sequence seems to converge to the point infinity, though infinity is not a point on the real line. We will see more about such sequences; these are sequences that diverge to infinity, we will study these sequences also.

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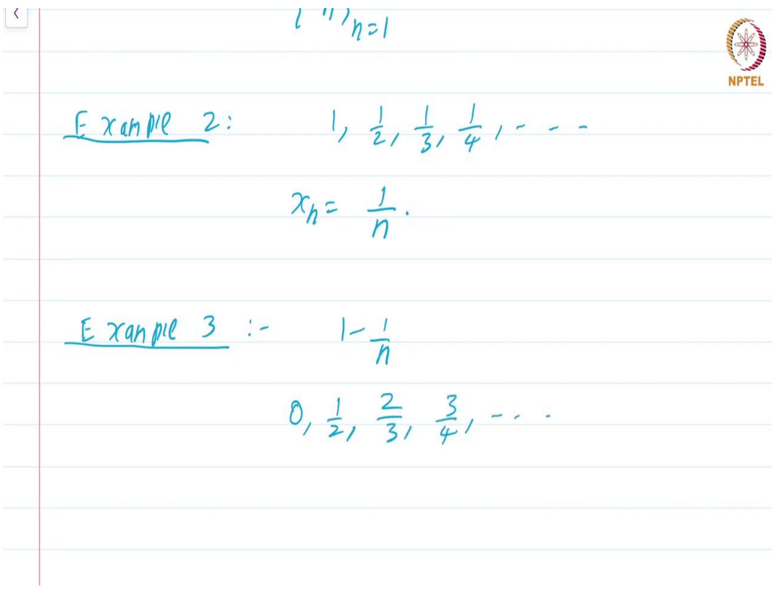

But that is not the central point of the course, so let me give another example which is more relevant; consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$, this is just the sequence $x_n = \frac{1}{n}$. Sometimes we shall be even more loose and just say the sequence $\frac{1}{n}$, sometimes you will just say the sequence $\frac{1}{n}$, ok. Now this sequence is ever decreasing; not only is it ever decreasing, it seems to be approaching or limiting to the point 0, the number 0, we will make precise what this means soon.

Let us just twist this example a little bit, let us consider $1 - \frac{1}{n}$, $1 - \frac{1}{n}$; the terms look like $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$ and so on. This is a sequence that is ever increasing and seems to converge to the point 1, ok.

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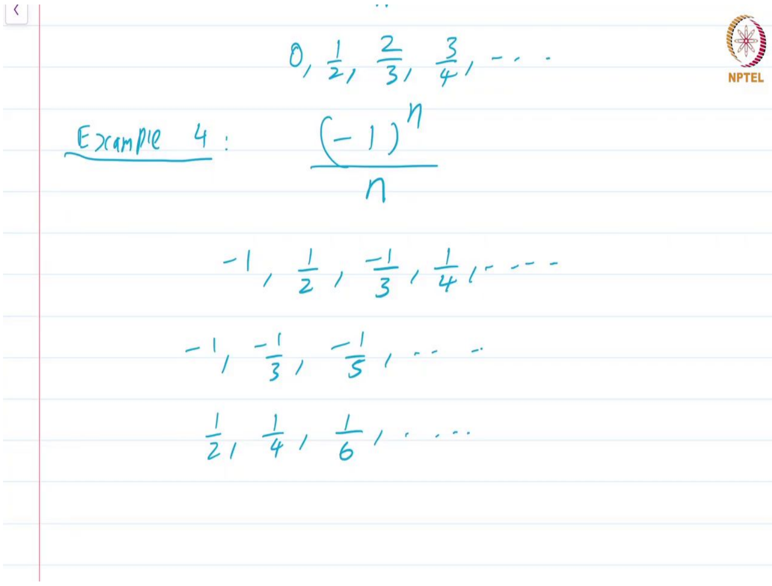

Now let us twist example 2 again and consider this sequence example 4, which is the sequence $\frac{(-1)^n}{n}$. How does this sequence look? Well the first term is $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \cdots$, and so on.

If you observe carefully, this sequence is neither increasing nor decreasing; it seems to be jumping here and there, but if you just collect together all the terms coming from the odd x_n 's, where n is odd, you get the sequence $-1, \frac{-1}{3}, \frac{-1}{5}, \cdots$ and so on. Whereas, if you collect all the even terms of the sequence; you get $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots$ and so on.

Now, a moment's thought will tell you that, both these encode subsequences seem to be approaching 0; and so does the main sequence minus $\frac{(-1)^n}{n}$, that also seems to be converging to 0, even though it is jumping around. One common mistake that students make in a first course in real analysis is to think that, if a sequence is convergent, it cannot jump around like this; it must either be increasing or decreasing at least after a point, that is not true.

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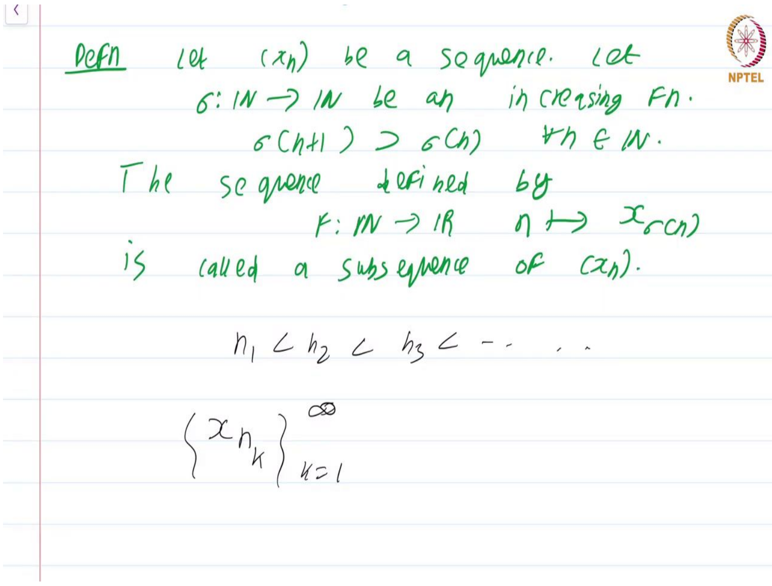

So, this example motivates another definition, let x_n be a sequence, let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be an increasing function. What does this mean? This means that $\sigma(n+1) > \sigma(n)$ for all $n \in \mathbb{N}$, ok. Then the sequence defined by $F : \mathbb{N} \longrightarrow \mathbb{R}, n \longmapsto \sigma(n)$ is called a subsequence, is called a subsequence of x_n.

So, if you think about this definition, all it is saying is that a subsequence is nothing but terms selected from a sequence, but in order; you can ignore some terms, but you will have to go it in the same order, you cannot change the order of the terms appearing in a subsequence.

So, another way to think of a subsequence is like this; you are given numbers $n_1 < n_2 < n_3 < \cdots$t. And you are just considering the new sequence $\{x_{n_k}\}_{k=1}^{\infty}$, where now k runs from 1 to ∞; the sequence is no longer indexed by n, but it is indexed by this n_k and k is running from 1 to ∞, ok. Now let us see some more examples, continue with the examples of sequences, at the same time see examples of sub sequences also.

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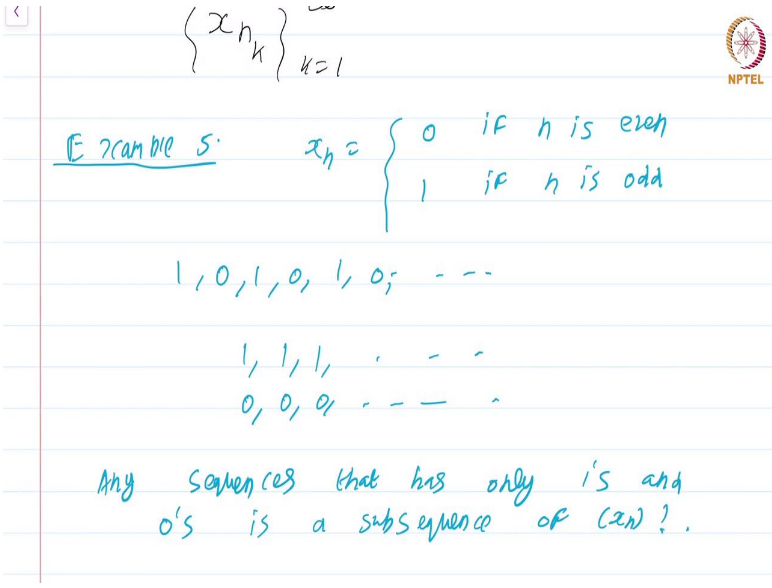

Example 5, what do you do is, you define x_n in a somewhat convoluted manner;

$$x_n = \begin{cases} 0 & \text{if n is odd} \\ 1 & \text{if n is even} \end{cases}$$

Or rather let me interchange, there is no particular reason why I am going to interchange, but it will be more useful;

$$x_n = \begin{cases} 1 & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

Ok. So, the sequence goes like this $1, 0, 1, 0, 1, 0, \ldots$. Again it has two interesting sub sequences $1, 1, 1, 1, \ldots$ and $0, 0, 0, 0, \ldots$. Note these are not the only sub sequences, these are not the only sub sequences; any sequence that has only 1's and 0's is a subsequence of this particular x_n.

Please check why that is true, but these two 1, 1, 1, 1, 1 and 0, 0, 0, 0, 0 are interesting sub sequences; these are known as constant sub sequences.

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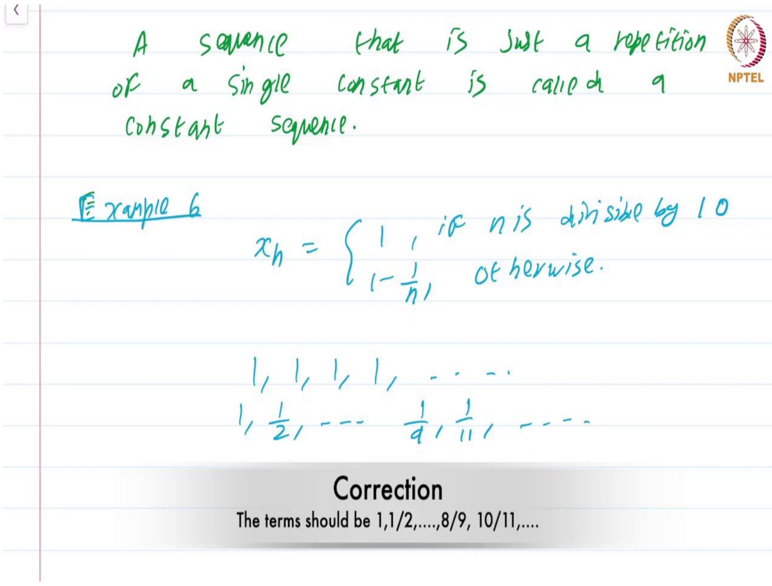

A sequence that is just a repetition, just a repetition of a single constant is called a constant sequence; not a very creative name, but at the same time no need to be creative when it is not needed, ok.

Now, let us see some complicated examples. I believe we are at example 6, we are almost there to our goal of 7 examples;

$$x_n = \begin{cases} 1 & \text{if n is divisible by 10} \\ 1-\frac{1}{n} & \text{otherwise} \end{cases}$$

Ok. This sequence also has some interesting sub sequences; one of them is 1, 1, 1, 1, 1, ok. Another interesting sub sequence is $1, \frac{1}{2}, \ldots, \frac{1}{9}, \frac{1}{11}, \cdots$ and no $\frac{1}{10}$, because it is 1, if n is divisible by 10. so on.

Now, this sequence looks to be convergent, because the terms seem to be decreasing; but every once in a while, there is a pesky one that shows up and ruins the day.

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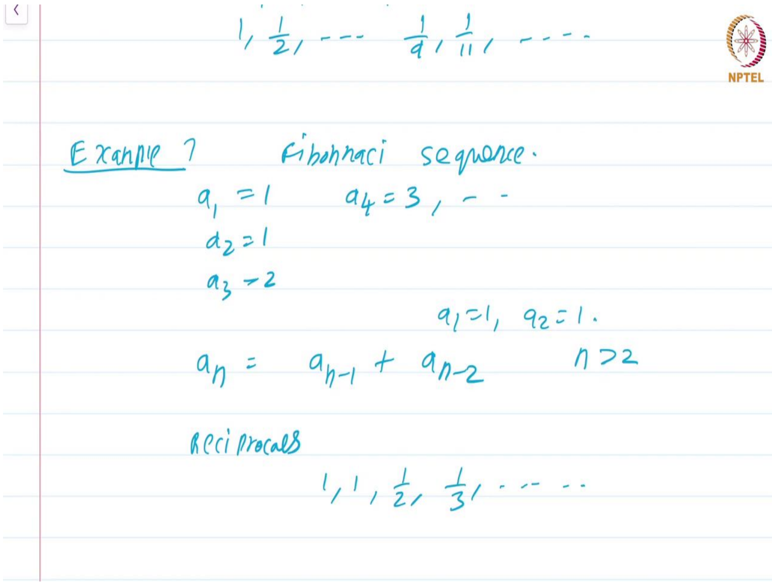

So, this sequence will actually not be convergent, the final example, example 7 is a very famous sequence, it is the Fibonacci sequence. How is this defined? Well you define $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3$and so on.

How does this go? Well $a_n = a_{n-1} + a_{n-2}$ and $n > 2$ for this definition to make sense; and we define $a_1 = 1$ and $a_2 = 1$, ok. So, it is a sequence that goes 1, 1, 2, 3, 5 so on. Clearly this sequence also seems to diverge to infinity, but we can form some interesting sequences from this Fibonacci sequence by considering reciprocals, we can consider the reciprocals; reciprocals will be $1, 1, \frac{1}{2}, \frac{1}{3}, \cdots$ and so on, ok.

And you can see that the reciprocals will sort of converge to 0. So, this concludes the set of examples; from the next module we will define convergence and we will solve many examples, find the sums, sorry find the limits, then find the sums of series, show that a particular sequence is not convergent, show that a particular sequence diverges to infinity so on and so forth we will solve plenty of problems. This is a course on real analysis and you just watched the module on the definition of sequences and examples.