Real Analysis - I
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Lecture – 6.4
Density of Rationals and Irrationals

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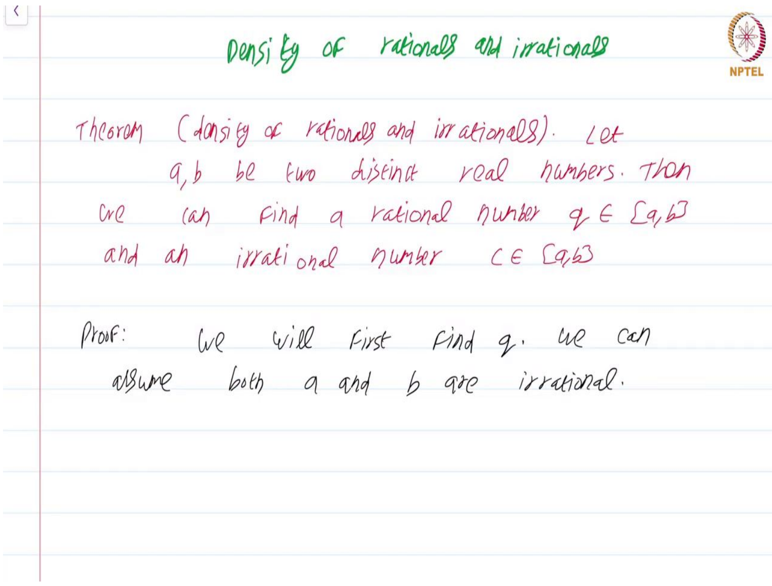

We have already seen that the real numbers are uncountable, that makes them very difficult to compute with many times. Fortunately, for all real-world applications, this uncountability is not such a big deal because you can always approximate a real number by rational numbers. This phenomenon is known as density.

It will be more clear why it is called density when we reach the chapter on topology, where I make precise what density is. But nevertheless, the statement I am about to state, the theorem I am about to state can be justifiably christened as density.

So, theorem, density of rationals and irrationals; what this theorem says is the following. Let a, b be two distinct real numbers. Then, we can find a rational number $q \in [a, b]$ and an irrational number $c \in [a, b]$. Proof ; we will first construct, we will first find q, ok.

Now, we can assume both a and b are irrational. Please pause the video and think about why we can make this assumption in this case. Hope you figured out that, if either a or b is rational, then you can just choose that to be the required rational number q ok.

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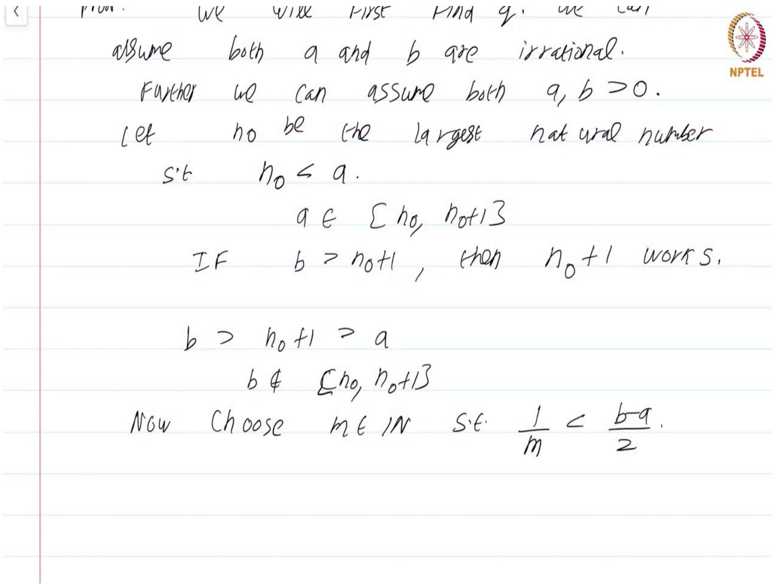

Further, we can assume both $a, b > 0$. I leave it to you to check what would happen in the other possibilities for a and b. I will just tackle this case; it really makes no difference.

Now, let n_0 be the largest, natural number such that $n_0 < a$, ok. In other words, $a \in [n_0, n_0 + 1]$, ok. Now, there are two possibilities. If $b > n_0 + 1$, then $n_0 + 1$ works, ok; because $n_0 + 1 \geq a$.

In fact, it is got to be greater than a, because we already assumed that a is irrational and it is less than b. So, we have found a natural number that lies in between a and b. So, we are done. So, we can assume that $b \notin [n_0, n_0 + 1]$. We can make this assumption ok. Now, choose m in the natural numbers such that $\frac{1}{m} < \frac{b-a}{2}$, ok.

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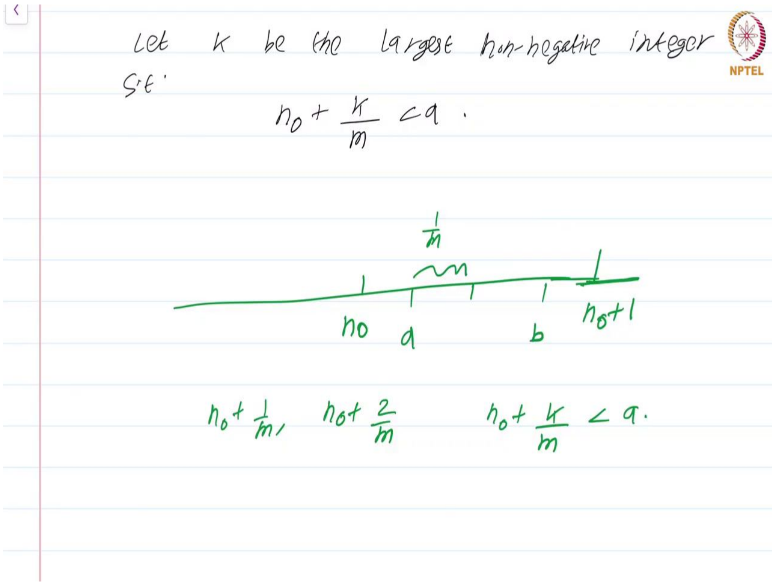

And let k be the largest number, largest natural number, again non negative. Let me just make that precise, non negative integer such that $n_0\frac{k}{m} < a$. So, a lot of confusing things have happened. So, let us draw a picture to make sense of what is going on. So, we have the points a and b. We know that n_0 is here; and under our hypothesis $n_0 + 1$ is here and $n_0 + 1$ exceeds b, ok.

Now, what we have done is we have chosen $\frac{b-a}{2}$ that is exactly half this length; and then, we are choosing a natural number so large that $\frac{1}{m}$, the size of $\frac{1}{m}$ is less than half the size of this interval $[a, b]$, ok. Then, what we are doing is starting from n_0, we are first going through $n_0 + \frac{1}{m}$, then we are going to $n_0 + \frac{2}{m}$ so on and so forth. And we are finding the largest integer such that $n_0 + \frac{k}{m} < a$, ok.

There will always be such an integer simply because $n_0 < a$. $n_0 + \frac{1}{m}$ could exceed a, in which case the choice of k would be just 0. Eventually, when k exceeds m you get $n_0 + 1$, right?. When $k = m$, you get $n_0 + 1$ which certainly exceeds a. So, at some integer, it has to be the largest integer such that $n_0 + \frac{k}{m} < a$. There will always be such an integer, ok.

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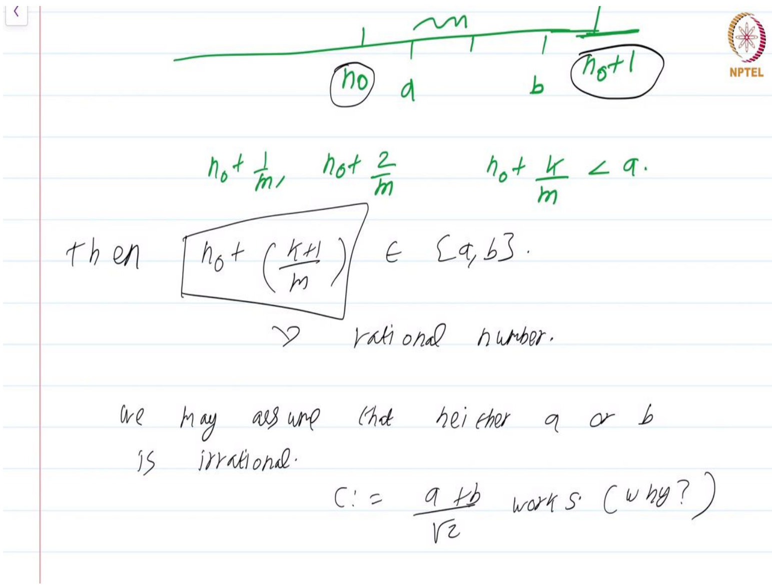

Now, n_0, we know that $n_0 + \frac{k}{m} < a$; that means, $n_0 + \frac{k}{m} \in [a, b]$, ok. Why is this the case? Well, because $n_0 + \frac{k}{m} < a$, $\frac{1}{m} < \frac{b-a}{2}$. So, adding $\frac{b-a}{2}$ to something less than a cannot exceed b. Simply, because the length of the interval $[a, b]$ is $b - a$ and $\frac{1}{m} < \frac{b-a}{2}$, ok.

Clearly, this is a rational number by construction. The trick of this proof is rather simple. By now, we are experts in trapping points in between some intervals. All we do is we trap the point a between n_0 and $n_0 + 1$. Once having trapped that, what we did is; we just essentially put a scale in between n_0 and $n_0 + 1$; then, made the rulings in the scale so small that we can literally mark off a point that lies in the interval $[a, b]$. That is all we have done ok.

Now, what about the irrational case? That is even simpler; we may assume that neither a or b is irrational, neither a or b is irrational, ok; for the same reasons that we made an analogous assumption. Then, c which is defined to be $\frac{a+b}{\sqrt{2}}$ works; ok. Why? I leave it to you to check why this works; it is rather straightforward.

So, this concludes the lectures for this week. We have seen quite a lot of properties of the real numbers. Next week, we will begin sequences. This is a course on real analysis and you have just watched the module on density of rationals and irrationals.