Real Analysis - I

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Lecture – 6.1
NIP+AP => Completeness

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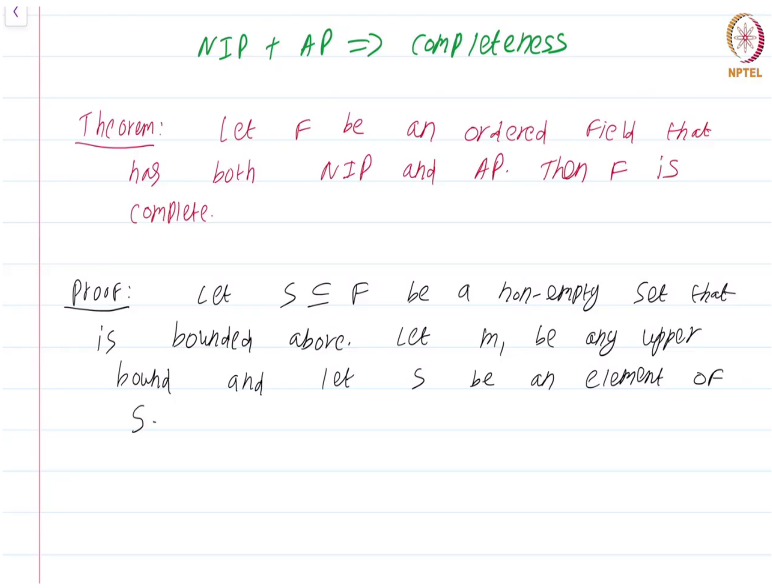

The aim in this module is to show that the nested intervals property together with the Archimedean property actually imply Completeness, without further adieu let me state the theorem. Let F be an ordered field that has both nested intervals property and Archimedean property. Then F is complete. What I mean by F has nested intervals property is that intersection of nested closed intervals is always non empty and what I mean by Archimedean property is the corresponding conclusion in the statement of the Archimedean property that we saw holds true for this field F.

Now, this theorem's proof is very important in the sense that the idea behind the proof makes it very clear why we need both nested intervals property and Archimedean property to ensure that there are no holes. What we are going to do is we are going to get better and better approximations of the required least upper bound for a set and show that the intersection of all these approximations is going to be the required least upper bound, that is the basic idea.

So, what we do is start with a set that is bounded above. Let $S \subseteq F$ be a nonempty bounded set, non empty set that is bounded above. We want to construct the required least upper bound. We proceed as follows, let m_1 be any upper bound, that is a good place to start and let $s \in S$, take any element in S.

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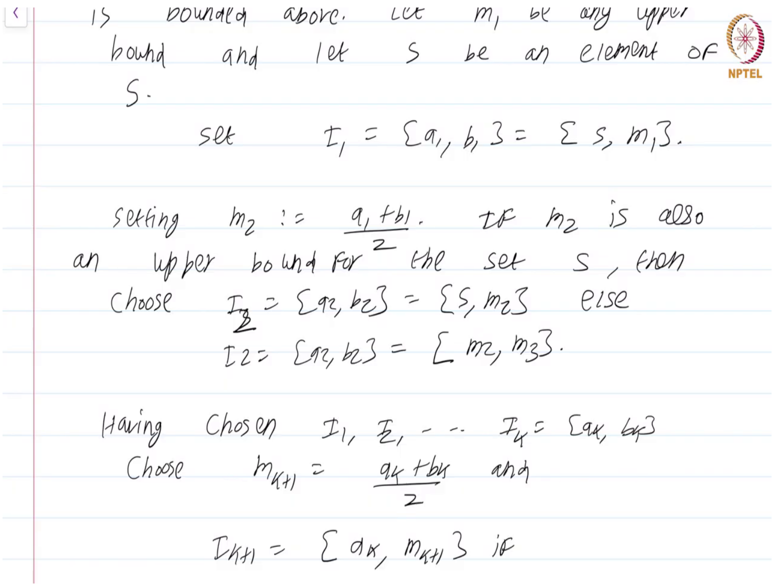

Set $I_1 = [a_1, b_1] = [s, m_1]$, the closed interval $[s, m_1]$.

Now, what is the rational or the logic behind this choice? Well, clearly if the set S has any least upper bound it has to lie in this interval $[s, m_1]$, no choice, s is an element of the set. So, it necessarily the least upper bound has to be greater than s or at least as large as s, greater than or equal to s and it has to be less than or equal to m_1 simply because m_1 is any upper bound, I mean some upper bound and the least upper bound is always less than or equal to any choice of an upper bound.

So, the required least upper bound has to be in this interval I_1. Now, we will make this approximation better by setting $m_2 = \frac{a_1+b_1}{2}$, look at the two endpoints and just take the mean of these two, $\frac{a_1+b_1}{2}$. Now, if m_2 is also an upper bound for the set S, then choose $I_2 = [a_2, b_2] = [s, m_2]$ else, $I_2 = [a_2, b_2] = [m_2, m_1]$.

What is the logic behind this choice? if m_2 is also an upper bound; that means, we have found an upper bound for the set S that is smaller than m_1 for sure. So, we have a better upper bound and if m_2 is not an upper bound for the set S, then we have moved the left end point of the interval we have moved it to m_2; that means, we have chosen an element that is closer to the required least upper bound. Essentially what we have done is irrespective of which choice we take for I_2, the size of I_2 is half the size of I_1. So, we have moved to a better approximation of the least upper bound ok.

Now, having chosen $I_1, I_2, ..., I_k$, $I_k = [a_k, b_k]$, choose $m_{k+1} = \frac{a_k + b_k}{2}$, and $I_{k+1} = [a_k, m_{k+1}]$ if m_{k+1} is an upper bound for S, else $I_{k+1} = [m_{k+1}, b_k]$. In this way we successively construct the required nested intervals.

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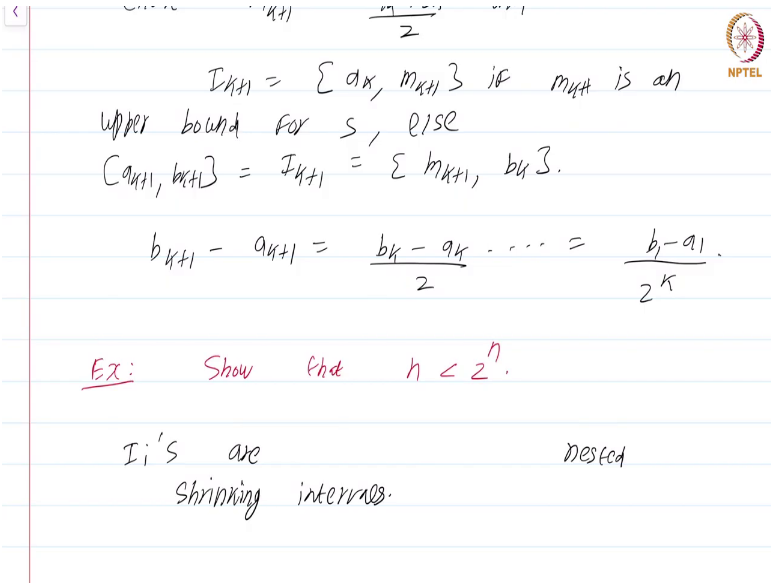

Now, observe that by very construction. So, this I_{k+1}, I will again call it $I_{k+1} = [a_{k+1}, b_{k+1}]$. By construction, you can prove this rigorously by induction if you want to, but it should be very obvious to you that, $b_{k+1} - a_{k+1} = \frac{b_k - a_k}{2} = \frac{b_1 - a_1}{2^k}$.

So, what has happened is at each stage, I_2 will be half the length of I_1, I_3 will be one-fourth the length of I_1 and I_4 will be one-eighth the length of I_1 so on and so forth. Here the length of the interval is just the right end point minus the left end point, the intuitive length of the interval. So, we have that these intervals are shrinking.

Now, it is an easy exercise for you to show that $n < 2^n$. This is a very easy exercise. You can show this by induction. Once you have that, we will have that these I_i's are a sequence of nested shrinking intervals, ok, let me not use the word sequence, I will just use I_i's are nested shrinking intervals ok.

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Shrinking intervals.

Therefore, by the nested shrinking intervals theorem, we must have a unique point in $\bigcap\limits_{i=1} I_i$, say u.

Claim : $u = \sup S$.

Suppose $s_0 \in S$. Each b_i, by our Construction, is an upper bound for S.
And assume $u < s_0$.

Therefore, by the nested shrinking intervals theorem, we must have a unique point in $\cap_{i=1}^{\infty} I_i$, say let us call this u, let us call the single point $u \in \cap_{i=1}^{\infty} I_i$.

Now, I want you to do the following. Pause the video look, through the notes where we proved the nested intervals theorem, the Archimedean property, the nested shrinking intervals theorem and so on and make sure you understand that both the nested intervals property as well as the Archimedean property are indeed being used to conclude that there is a unique point u in the intersection.

So, in this one statement I am actually compressing several steps. I am leaving it to you to unwind these steps. It is very important that you do this exercise, please do that ok. So, at any rate we have found a point u. Claim is that $u = \sup S$. How are we going to show this? Well, we use the fact that our choice of I_i's were not arbitrary, but we were making better and better approximations, how do we do that?

Suppose, $s_0 \in S$. Now, first of all observe that each b_i, by our construction, is an upper bound for S, that is how we constructed each one of the I_i's. We ensured that the right end point is always going to be an upper bound of S. That at each stage we had two possible choices, we always chose that possibility that ensures that the right end point of each one of these intervals is always an upper bound, ok.

So, we have chosen the right end point as always an upper bound. Now, suppose $s_0 \in S$ and assume $u < s_0$. We have to show that u is a least upper bound, first of all let us see that u is in fact an upper bound. We have chosen $u < s_0$, we want to reach a contradiction to this.

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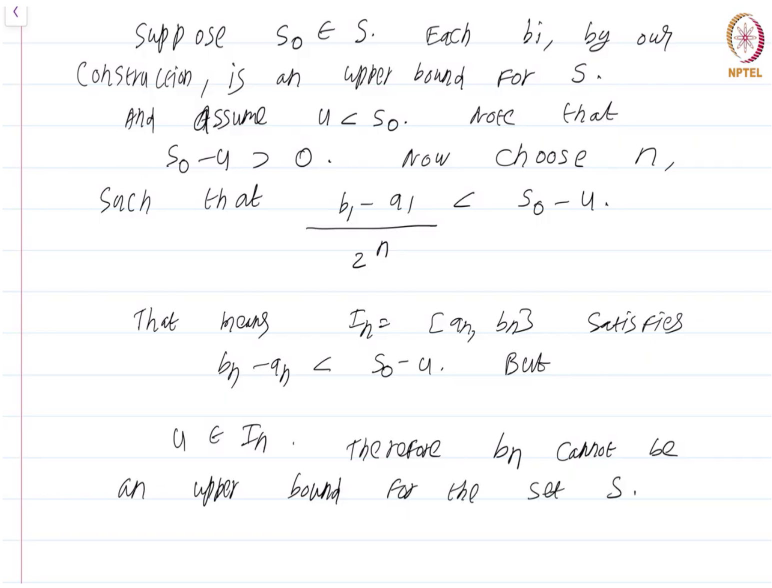

Now, note that this means $s_0 - u > 0$. Now, choose n such that $\frac{b_1 - a_1}{2^n} < s_0 - u$. Again note that to choose such an n you have to use the Archimedean property. Please pause the video, this is a subtle point, make sure you understand how the Archimedean property comes into the picture here. First of all, the fact that $n < 2^n$ is used and then you use the Archimedean property to construct n so large that $\frac{b_1 - a_1}{2^n} < s_0 - u$.

Now, that means $I_n = [a_n, b_n]$ satisfies $b_n - a_n < s_0 - u$, ok, but, $u \in I_n$. In fact, u is the unique element in the intersection $\cap_{i=1}^{\infty} I_i$, not just this particular one. Therefore, b_n cannot be an upper bound for the set S. Why is this the case? Well here at this juncture let me just draw a picture.

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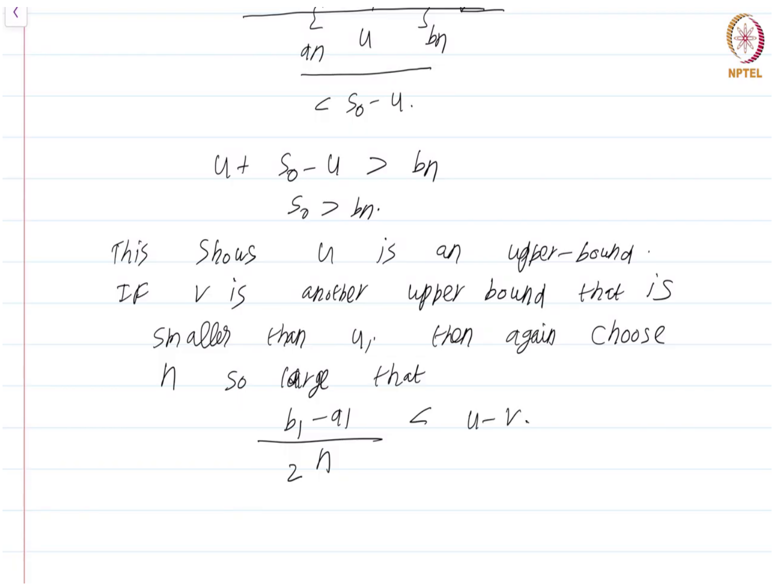

We have this interval $[a_n, b_n]$ ok, we have this point u which is somewhere inside, we know that $b_n - a_n$, this length is certainly less than $s_0 - u$, ok. That means, $s_0 \notin [a_n, b_n]$ because the entire length of this interval $[a_n, b_n] < s_0 - u$. So, $u + s_0 - u > b_n$ because the length of this interval is itself just maximum going to be $s_0 - u$, right?, in other words $s_0 > b_n$, ok.

So, this shows u is an upper bound, u is an upper bound. What remains to be shown is that it is the least upper bound. So, if v is another upper bound that is smaller than u, then we apply the same trick in a different manner. Then again choose n so large that $\frac{b_1 - a_1}{2^n} < u - v$.

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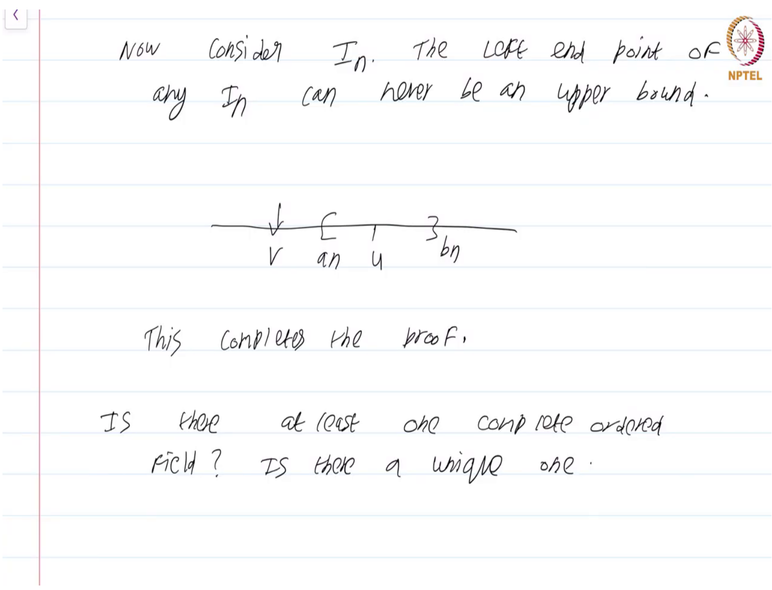

Now, consider I_n. Again if you carefully look at our construction the left end point of any I_n can never be an upper bound, can never be an upper bound, that is exactly the way these intervals I_n were constructed.

The right end point will always be an upper bound, the left end point will never be an upper bound, that is exactly how this was constructed, ok. Please check that. Once you have that the left endpoint cannot be an upper bound, the same picture with the slightly different orientation will finish the proof.

We have $[a_n, b_n]$, again we have u here and we know that this n has been chosen so large that the size of $[a_n, b_n]$ which is same as $\frac{b_1 - a_1}{2^n} < s_0 - u$, which means that v has to be to the left of $[a_n, b_n]$, again this is the same logic, this completes the proof, this completes the proof.

We have shown that this u has both properties of a supremum, it is both an upper bound and it is also the least upper bound. So, we have shown that nested intervals property plus Archimedean property actually implies completeness. Now, why is this very crucial because if you carefully look at the proof, we had used nothing basically to show that you can approximate the least upper bound in the first step, we just shows one point in the set and chose one upper bound, we know that this least upper bound if you take this is has got to be there.

Then in the second step we again chose a better, better in the sense that the length of the interval is becoming half, better approximation. The crucial point is the Archimedean property shows that these approximations get better and better and better.

In other words what is really crucial in this proof apart from the nested intervals property is the fact that $\frac{1}{n}$, the quantity $\frac{1}{n}$ can actually be made as small as possible. In just a few modules down the line, as part of next week's chapter on sequences, you will see that the Archimedean property actually implies that $\frac{1}{n}$ converges to 0. Do not worry if you do not know what converges means, it will be made clear in the next chapter.

So, the Archimedean property makes $\frac{1}{n}$ close to 0 and it is the Archimedean property that in fact says that as you keep increasing this n, the approximations become better and better and better. So, the way we have used these in tandem is to use the nested intervals property to produce the least upper bound, but you can apply the nested intervals property in the first place to get a unique point solely because the Archimedean property allows you to keep improving the approximations.

So, this hopefully clarifies the remark towards the end of the last module that we need both, the nested intervals property as well as the Archimedean property to ensure that there are no holes. So, essentially the Archimedean property is used to surround the hole by really close approximations. You will always be able to approximate a hole just with the nested intervals property. nowhere do I actually use the Archimedean property to construct these I_n's, but the fact that they sort of 0 in on the hole is made precise by the Archimedean property.

So, I hope this theorem, even though it is not really going to play any role in this particular course, clarifies completeness to a reasonable extent. Now, what have we done? We have shown that a complete ordered field has no holes. But several questions remain. Our aim was to produce an appropriate ordered field that models the real line. Question is, is there at least one complete ordered field? Right, all these theories will be a waste if there is no complete ordered field, second is there a unique one is there a unique one ok. The answer to both questions is the following theorem which I will not prove.

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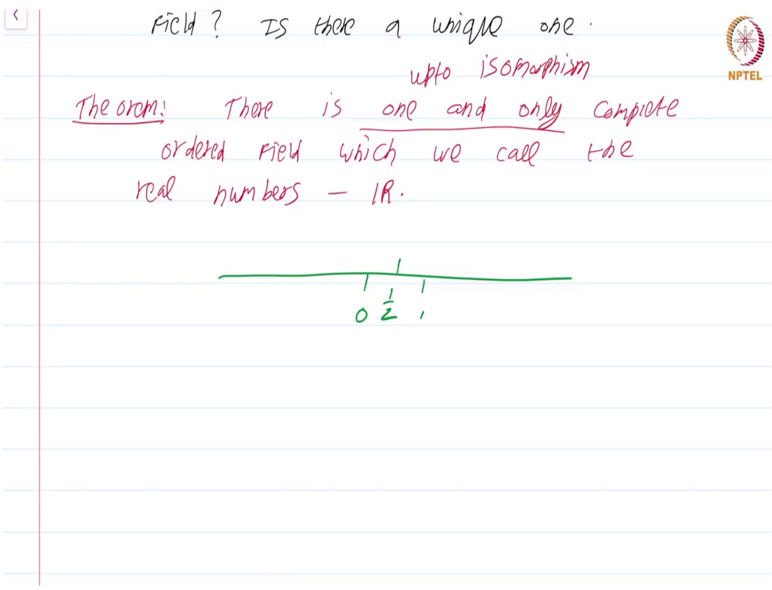

Let me, after stating the theorem, illustrate why I would not prove this theorem. There is one and only one complete ordered field which we call the real numbers, real numbers $\mathbb{R}$ ok. When I say one and only I must be precise up to isomorphism, I will not make precise what isomorphism is. If you have taken a course on linear algebra or abstract algebra probably it is clear to you what this isomorphism means.

So, there is essentially only one complete ordered field which we are going to call the real numbers ok. Now, why am I not going to prove this theorem because the proof is really long.

In fact, if you want to write a complete proof of this it will take a small textbook, indeed there are many, the notes refers to my favorite one that gives the complete proof and it really does not clarify anything that is needed for later analysis. This is one of those theorems that needs to be proved for the theory to have any validity. If there is no complete ordered field the whole theory is nonsense. So, you need to prove this theorem, but the details of the proof are fairly boring that there is no point in engaging in this activity right now.

Moreover, once you take an abstract course on metric spaces you will be able to show this in a much easier manner. So, I am not going to bother proving this now, I am going to leave this proof to you. If you are interested you can look at the references in the notes ok. One final point to address. We have produced the complete ordered field it certainly does not have any holes, but our aim at nothing to do with completeness or ordered fields or anything right at the beginning, all we wanted was we had this real line, when you put the rational numbers in the real line there are holes, we want to plug these holes right?. Whether the final structure that plugs these holes is an ordered field or a complete ordered field or it is a topological space or it is a Donkey is completely irrelevant to us, we want to somehow fill all these holes.

Now, we have filled all these holes, but how do you know that what you got in the end, the real numbers has anything to do with the line that you started off with. We have done, in fact, I have said that the construction is long and hard and I am even skipping the construction. Why should we have the real numbers that satisfy all these axioms, why is it not a plane or a sphere or some other complicated object, why is this exactly the real line. Well that is something that I do not want to get into. Again please check the note for references.

To show that the real numbers correspond to the real line, I mean I should not really be using the real line at all, because I am assuming what I want to show as write implicitly in my language itself, to show that the real numbers model a straight line the first step to understand what a straight line is precisely so that you can show that both are same ok.

Now, it turns out the correct way to do this is to axiomatize Euclidean geometry. Here we go again. I will not bore you with another set of lectures with axiomatization. So, do not worry.

So, you will have to model the Euclidean geometry that you have studied in school via axioms. It will turn out under a particular axiomatization which makes perfect intuitive sense, the real numbers do correspond to points of a straight line. For more details please check the notes for references. This is a course on real analysis and this is the module titled NIP plus AP implies Completeness.