Real Analysis - I
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Lecture - 1.2
Why Study Real Analysis

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Why bother studying Real Analysis? The tools of calculus are used throughout the natural sciences, engineering and humanities. Therefore, an error in an application of a tool of calculus could have catastrophic consequences.

The collapse of a suspension bridge or the Mars rover getting stuck inside a Martian ditch; both are possible. However, why should a person who is merely going to use the tools of calculus worry about the theoretical underpinnings of calculus? In this module let me try to convince you that a study of the tools of calculus is incomplete without going into the nuts and bolts.

So, the first reason is the fact that you want to ensure that your application of the tool is absolutely correct. But this is not much of a problem. All one has to be is be very very conscious when applying a tool, make sure that all the hypotheses are satisfied by the situation at hand, one need not really bother about why the theorem is true to actually apply it. But still there is some merit in studying how the tool works. So, let me list one reason, there is philosophical satisfaction, philosophical satisfaction in understanding why things work and not merely how to apply a tool.

There is a philosophical satisfaction in understanding why tools work. So, this is one reason why even if you are interested nearly in applications, it's good to have an understanding. But apart from that there is a deeper reason, you might be able to tweak you might be able to modify or tweak a tool to suit your new situation, suit a new situation.

It is not true that all the great tools of calculus have been discovered by mathematicians; many of the tools have been discovered by engineers who needed a new version of a particular tool or a slightly different version of an existing tool, knowing the background theory is of great use when you are trying to redesign an existing tool okay.

These are two good reasons why one is interested in a deeper study of calculus, why is somebody interested in the theoretical underpinnings. But, you will soon notice that trying to study real analysis in any depth requires you to pause and prove many results that are geometrically obvious.

You might sympathize with studying the deep and subtle results of real analysis, but why should you bother studying about those obvious geometrical geometrically obvious results? Why should you waste time proving those? So, let me illustrate with an example of a result that you are probably familiar with from a basic undergraduate calculus course.

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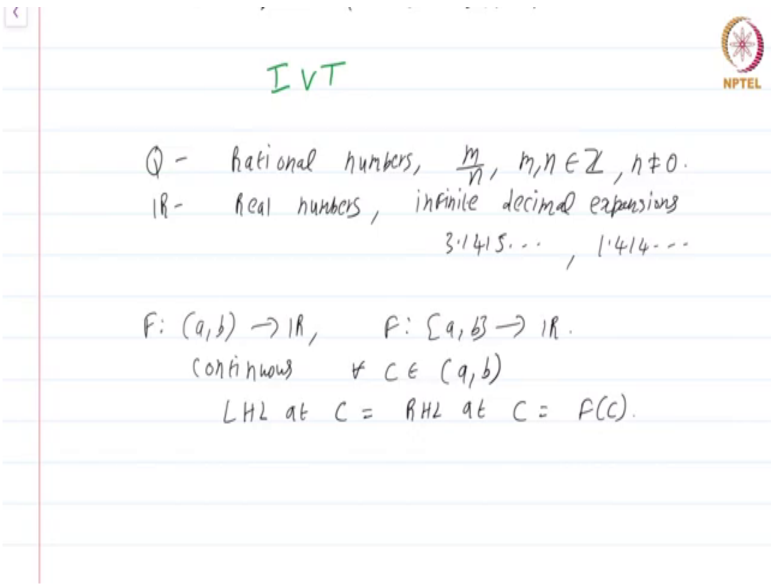

So, this is called the Intermediate Value Theorem which I will just abbreviate as IVT. So, for this I need to recall some notation from high school mathematics, first $\mathbb{Q}$ is the rational numbers. This is the set of all numbers of the form $\frac{m}{n}$ where m and n are integers; which are denoted by $\mathbb{Z}$ and n is not 0, the denominator is not allowed to be 0. Then $\mathbb{R}$ is the set of real numbers. So far you are used to thinking of real numbers as infinite decimal expansions in high school.

So, I will continue with that infinite decimals expansions; infinite decimal expansions. So, examples of real numbers are 3.1415, I do not know the digits of phi beyond this and 1.414… so on and so forth. So, these are examples of real numbers; they are infinite decimal expansions. So, in your high school and probably in an introductory course on calculus, you must have studied continuous functions.

So, we define continuous functions either from open interval (a, b) to $\mathbb{R}$ or from closed interval $[a, b]$ to $\mathbb{R}$, the definition is only slightly different, let me just read the open interval case and leave it to you, you definitely know the definition even in the close interval case.

A function is said to be continuous if for all $c \in (a, b)$, let me just use the open one for convenience for all $c \in (a, b)$ for all choice of points in the interval (a, b), we have the left hand limit at c is equal to the right hand limit at c is equal to the functional value at c, right.

So, this captures the intuitive fact that a continuous function cannot jump. The same definition holds for closed intervals except at the end points a and b, one of the limits either the left hand limit or the right hand limit doesn't make sense, same definition will work. Now, the intermediate value theorem captures the fact that a continuous function cannot jump in the following manner.

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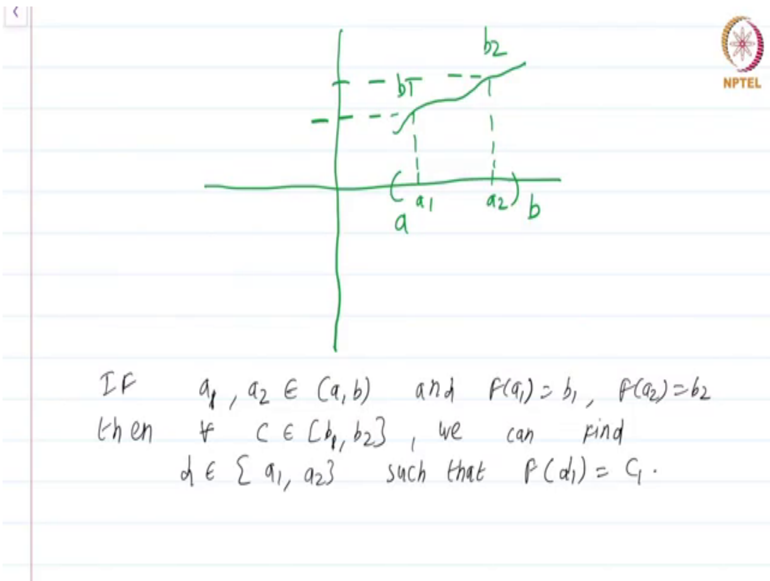

Let us draw a picture, here we have the x-y plane ,we have the points a, b, now the graph of the function will look something like this, okay. There are no jumps because I am graphing a continuous function. What the intermediate value theorem says is the following. If a_1 and a_2 are in (a, b) and $F(a_1) = b_1$, $F(a_2) = b_2$ then for all c that lies in between b_1 and b_2, we can find $d \in (a_1, a_2)$ such that $F(d) = c$.

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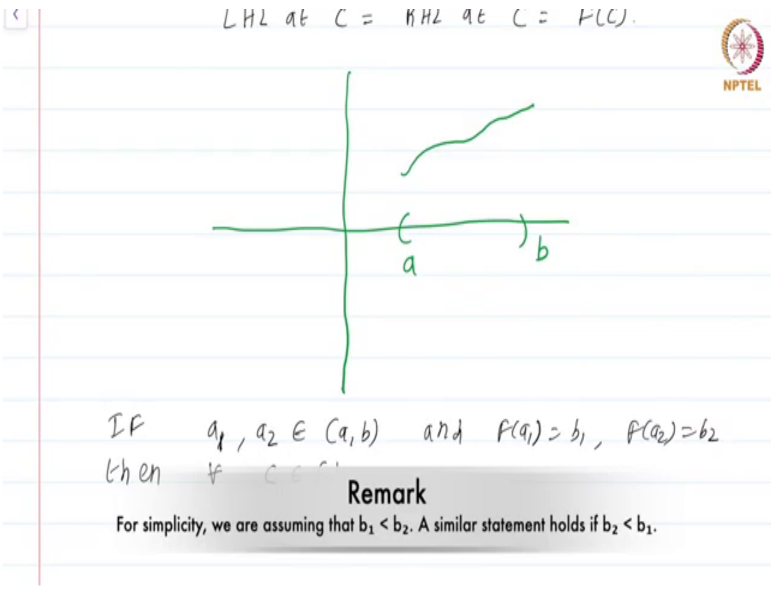

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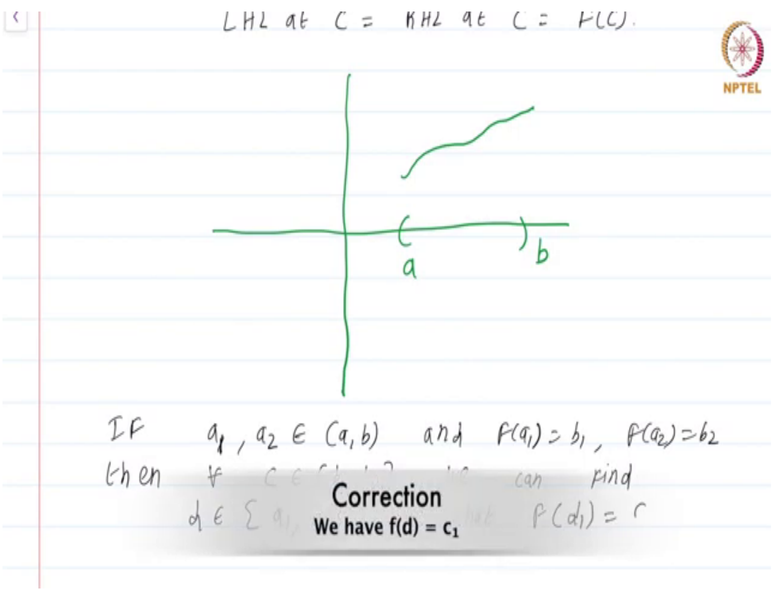

So, what the intermediate value theorem is saying is that if you choose two points a_1 and a_2 in this interval, look at the corresponding values which we called b_1 and b_2, look at the corresponding values. Then, all values that lie in between b_1 and b_2 are taken by this function by some point that lies in between a_1 and a_2. So, this is capturing the fact that the graph cannot just jump it as to move continuously from b_1 to b_2, okay. Now, this is one of the results that seem obvious. In fact, many would say that it is completely uninteresting to prove this result and there is no subtlety involved; it is a geometrically obvious result.

But, the apparent lack of subtlety is just an illusion; let me illustrate with a couple of examples. If you claim that the intermediate value theorem is utterly obvious and requires no proof, then I will pose a challenge to you.

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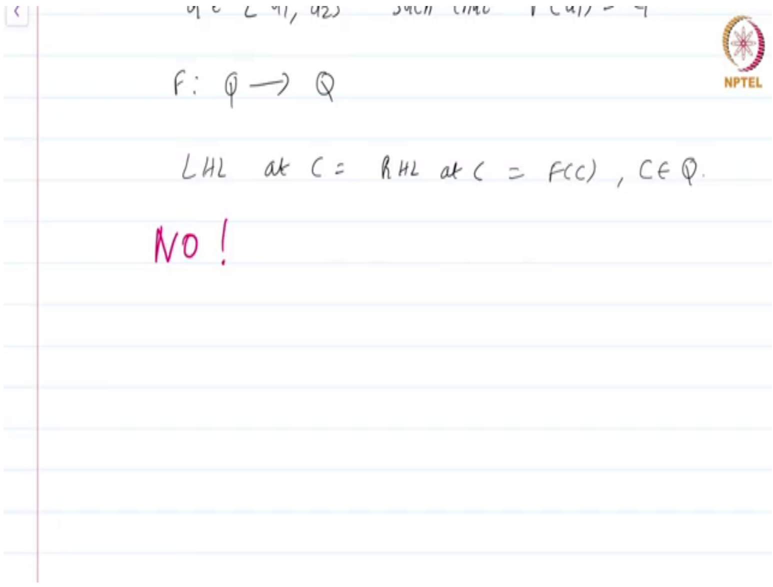

Suppose I give you a function F from $\mathbb{Q}$ to $\mathbb{Q}$ from rational numbers to itself and suppose I tell you that this function is continuous well what is continuity in the rational numbers you might ask? Well why not reuse the same definition? You can say that a function F from $\mathbb{Q}$ to $\mathbb{Q}$ is continuous if the left hand limit at c is equal to the right hand limit at c is equal to the functional value at c except c now belongs to $\mathbb{Q}$.

So, for all rational points, the left hand limit is equal to the right hand limit is equal to the functional value, that is the definition of a continuous function. Well, that being the case let us check what would happen in relation to the intermediate value theorem.

Is the intermediate value theorem true for such functions? I urge you to pause the video and think for a couple of minutes on this question; hopefully you thought about this question, the answer is surprisingly no. The intermediate value theorem is not true for functions from $\mathbb{Q}$ to $\mathbb{Q}$ even if they are continuous. Well, the only way you can show something is not true is by producing an example. Let us produce an example of a function that is continuous from $\mathbb{Q}$ to $\mathbb{Q}$, but nevertheless does not satisfy the intermediate value theorem.

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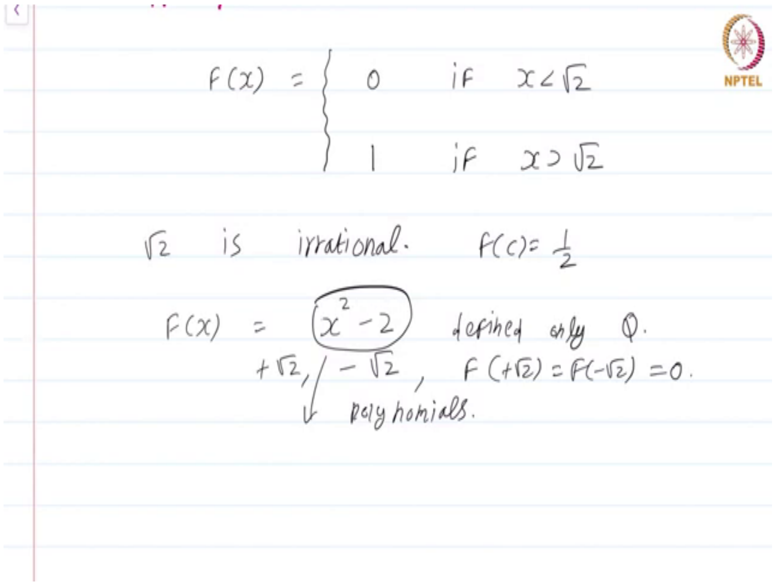

The function is defined as follows; $F(x)$ is defined to be 0 if $x < \sqrt{2}$, it is 1 if $x > \sqrt{2}$. Note, I have not defined what F is at the point $\sqrt{2}$, but I do not need to because $\sqrt{2}$, as you know from high school is irrational, it is not a rational number.

So, this function is certainly defined for all rational numbers in $\mathbb{Q}$, but the function clearly does not satisfy the intermediate value theorem; there is no c there is no c such that $F(c) = \frac{1}{2}$, there is no such c. So, the intermediate value theorem fails for this function.

You might object saying, why is this function continuous? Well, it is because the left hand limit is equal to the right hand limit is equal to the functional value at all points except possibly $\sqrt{2}$ but at $\sqrt{2}$ there is no issue because $\sqrt{2}$ is not a rational number. So, this function if you take the definition of a continuous function as one for which the left hand limit is equal to the right hand limit is equal to functional value there is no issue.

This function is perfectly continuous, but yet does not satisfy the intermediate value theorem. Another objection can be made. The definition of continuity that we have is not applicable to the rational numbers, simply because if you graph this there is a clear jump. Even though the jump occurs at an irrational number you might object saying that this function is not really continuous, I would not want to formulate continuity in this manner. Then there is now another example I can give you.

Look at the function $F(x) = x^2 - 2$ defined only on $\mathbb{Q}$, that is, I am not defining this function for the whole of $\mathbb{R}$ but I am restricting this function only to the rational numbers. Well, is this function ever going to be 0? This function will be 0 precisely at the points +$\sqrt{2}$ and -$\sqrt{2}$, right? At these points you have the value 0, at other points it is not going to be 0 of the real numbers, not just rational numbers.

This function $x^2 - 2$ is a polynomial, this is a polynomial. There is no controversy if I say polynomials should be continuous, defined on $\mathbb{Q}$. The last function that I defined I was I defined it in pieces. So, you might object saying it is not really a continuous function, but no such objection can be made to this function which is a polynomial.

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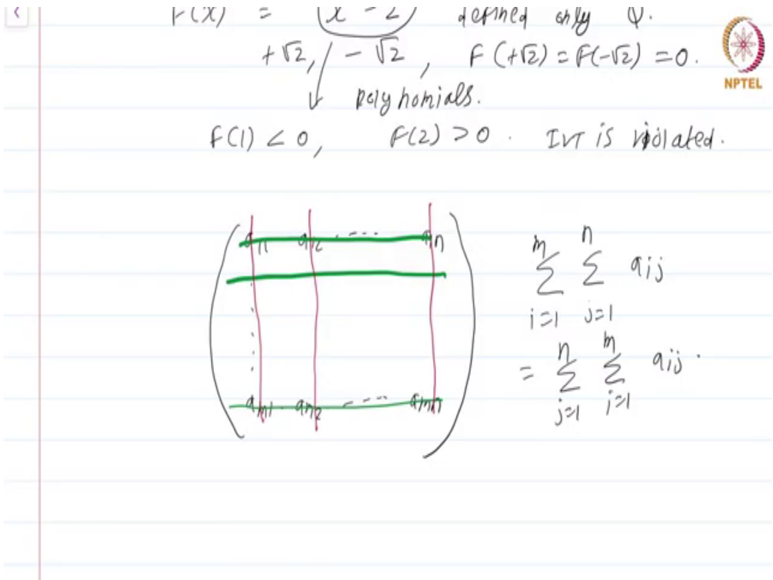

Well, since the value 0 is taken only at +$\sqrt{2}$ and -$\sqrt{2}$ and this function F is certainly negative when at the point 1. And at point 2 it is certainly positive, this function misses the value 0 even though it takes both a negative value and a positive value. So, the intermediate value theorem is violated; intermediate value theorem is violated.

So, this function illustrates that the intermediate value theorem is not true for rational numbers. So, the goal is to capture what does continuity really mean? As we have seen the properties of continuous functions not only depend on the function itself, but also on where they are defined; the so-called topological properties on the domain of definition.

If you want to clarify such concepts of continuity there is no choice but to engage in a deeper study of the real line and the deepest study of continuity that ends up being nothing but real analysis, okay. So, I have illustrated something as simple as the intermediate value theorem is not obvious, there is some deep subtlety going on; we need to know what is it about the real numbers that makes the intermediate value theorem work. But $\mathbb{Q}$ is lacking that feature. This key fact is known as completeness and just treating real numbers as infinite decimals does not really clarify what is it that $\mathbb{R}$ is having extra over $\mathbb{Q}$. So, we will study completeness in great detail next week, before that I would want to give one more example illustrating how casual use of knowledge of calculus can lead to profound errors.

So, let me illustrate with an example, let us take a matrix. A matrix is just a square array of numbers. So, let us just take,

$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ . & \dots & . \\ a_{m1} & \dots & a_{1n} \end{pmatrix}$, an m cross n matrix.

Now, suppose I want to sum all the entries of this matrix, there are two ways by which I could proceed. I could first sum up the first row then the second row so on and then the last row then add all these together or alternatively I could sum up the first column the second column so on and so forth till the last column and then add all those together.

So, obviously, it does not matter how you do this addition whether you do rows first and then sum up the results or columns first and then sum up the results, you are going to get the sum of each entry of the matrix; that you can write in as saying summation n equals, let me not use n, take $\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}$. So, all this says is that you can interchange the two summations.

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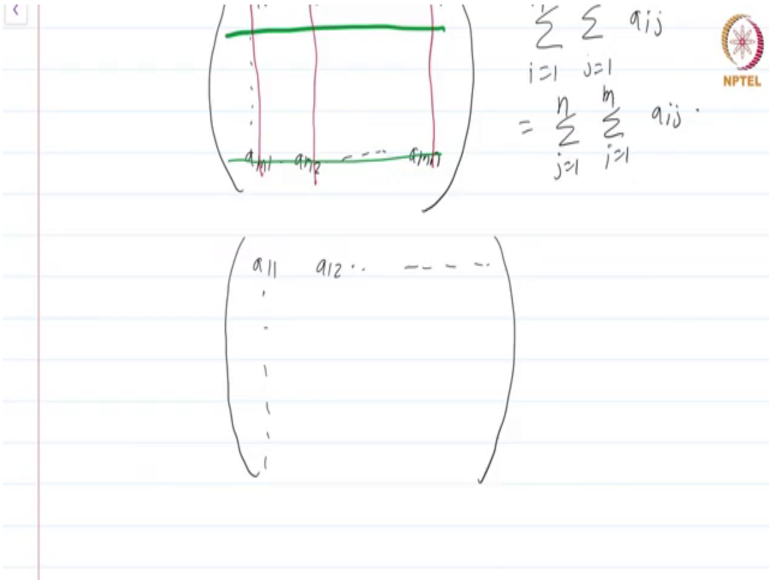

Instead, suppose I had an infinite matrix; that means, I have entries that go in both directions all the way. Suppose, I want to sum up all the entries in this matrix and the first question is what is the meaning of the sum of an infinite number of numbers? Well, you all have an intuitive notion of what it is from your high school mathematics; let us see what happens with an example when you do this summation in two different ways.

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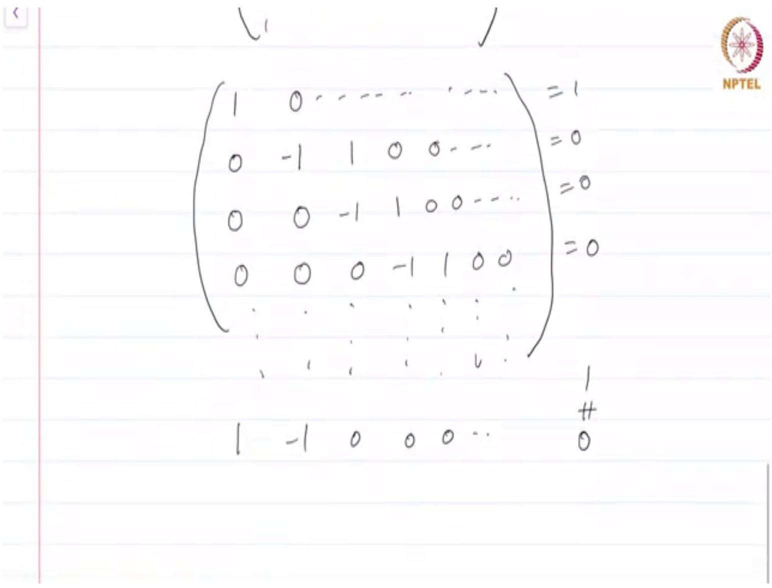

So, what we do is we consider a matrix, the first entry is 1, the second entry is 0 and its all 0s then its 0 here, -1 here so on and so forth, okay; then its 0 0 -1 so on and so forth, okay. So, you see the pattern of what is happening you have 1 0 0 0 0 you have 0 -1 so on, just once again let me just make one slight tweak.

Let me just make one slight tweak to this, let me put a 1 here, then let me put a 1 next to this. Now the pattern is clear, pattern is clear; the next one would be so on. Now I have put a last 0 here that really does not make sense because it goes on indefinitely.

Now, let us sum up by the rows. The first row sums up to 1, the second row sums up to 0, the third row sums up to 0 so on. When I put dots I must clarify again when I put dots I mean 0 0 0 here also its 0 0 0 and same thing here and this pattern repeats indefinitely, okay.

So, clearly the sum of the second row is 0, the sum of the third row is 0, sum of the fourth row 0 so on and so forth, okay. So, the sum of these rows if you see is just 1, its just 1, okay. Now on the other hand if I sum up by columns what happens? I get 1 for the first column, I get minus 1 for the second column, I get 0 later.

When I sum all these together I get 0, which is not the same as 1. What this shows is that summing up rows first and then summing up columns do not yield the same value; something goes wrong; it is not true that you can interchange the summations, okay. So, there is something deep going on and the only way to see what is it that goes wrong here is to understand what infinite sums really mean.

To give a precise definition of infinite sums and see that in many cases infinite sums do not behave the same as usual finite sums. Some of the laws such as commutativity and associativity mean need not hold for infinite sums, unless some conditions are put. This is a course on real analysis. And you have just watched a module on why study real analysis.