Real Analysis - I
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Lecture – 5.1
The Completeness Axiom

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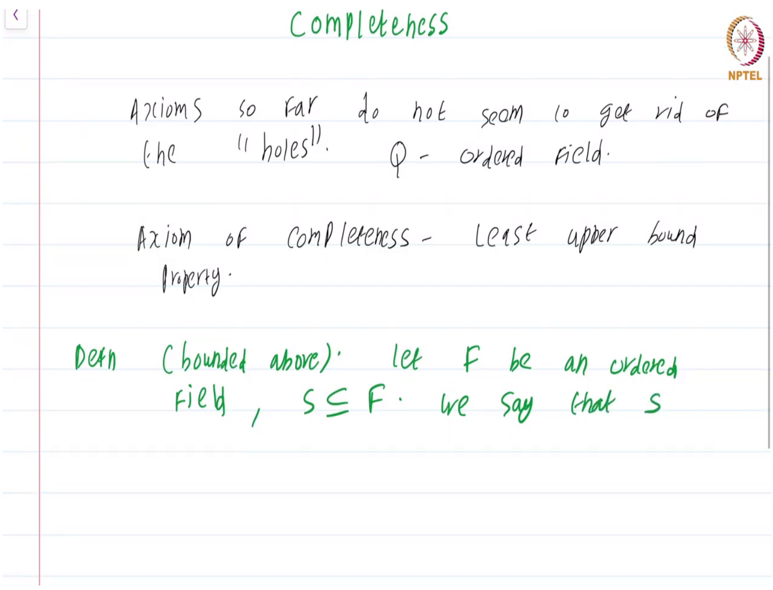

We now come to the crucial property that the real numbers possess, that rational numbers do not. First of all, note that our axioms so far do not preclude the possibility of holes, do not seem to get rid of the holes. So far the field axioms and the order axioms very much allow $\mathbb{Q}$; $\mathbb{Q}$ is an ordered field. The only thing we have seen is that every ordered field also contains a copy of $\mathbb{Q}$.

Now, the question arises what is the crucial property that $\mathbb{Q}$ is lacking that makes the appearance of these holes? What we need is the axiom of completeness. This axiom of completeness plugs all the holes as we shall see. Now, this is the central point of real analysis. What I am about to do now is what is the key that allows us to define limits and continuity and so on in a satisfactory manner.

Because of the central nature of this axiom, we shall present multiple versions of it. We shall spend some time on this and even in a later chapter on sequence and series we will visit this yet again. So, the most common way to state the axiom of completeness is through the least upper bound property. I would not say this is the simplest way to do it, but this is the most common way and to do that we need some definitions.

Definition, this is the definition of bounded above. Let F be an ordered field and $S \subseteq F$ ok. We say that S is bounded above if there is some $b \in F$ such that for all $s \in S$, $s \leq b$. So, there is some element in the field that dominates every single element from the set S. In that event we say that the set S is bounded above.

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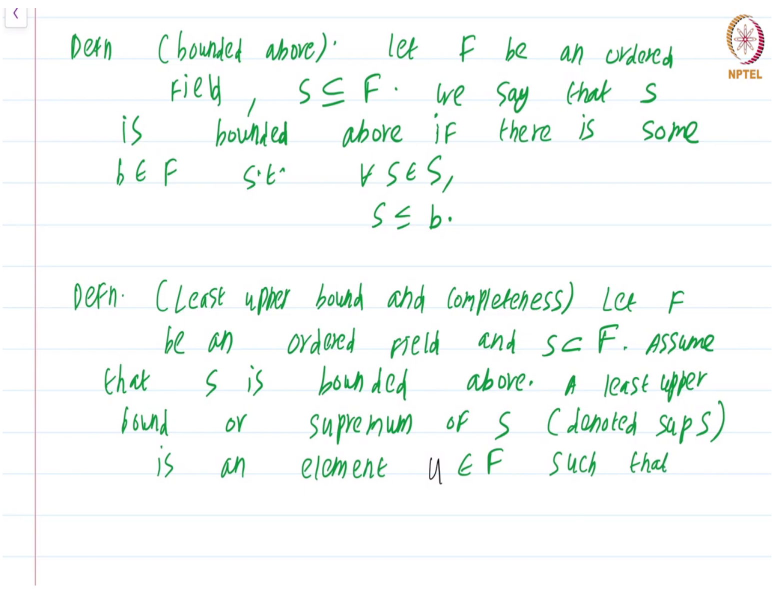

Now, the crucial definition is of least upper bound and completeness which is, as I have emphasized several times already in the short span, the central point the central key axiom that makes analysis possible. Again, let F be an ordered field and $S \subseteq F$, ok. Assume that S is bounded above; assume that S is bounded above.

A least upper bound or supremum of S, this is usually denoted this is denoted $\sup S$, is an element, is an element $s \in F$. Note, the element s must come from F. So, let me just not use s, let me just use u for upper. It is an element $u \in F$ such that property (i), u is an upper bound upper bound for S.

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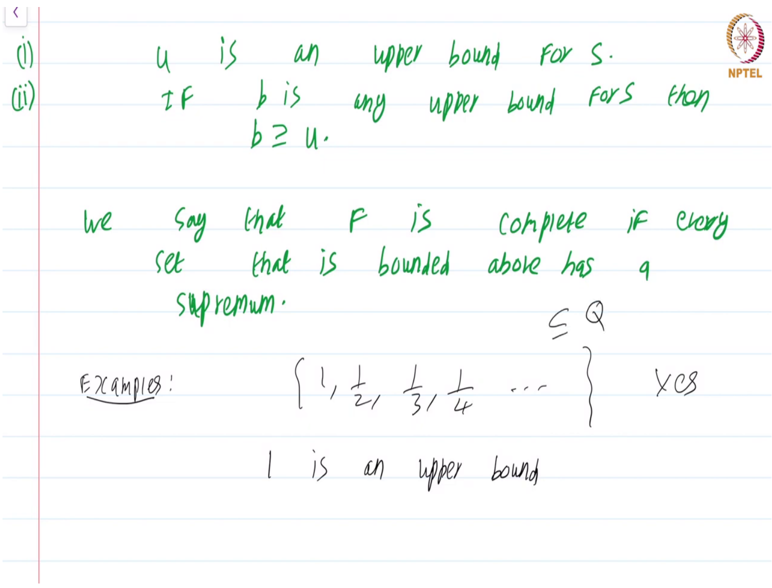

Well, that seemingly straightforward because we are trying to define what a least upper bound is the very least it can be is that it is an upper bound.

Property (ii) is the crucial property. What it says this is, if b is any upper bound for S, then $b \geq u$. In other words, this u is the least upper bound as the definition was trying to tell. We say that F is complete if every set that is bounded above has a supremum or least upper bound. That is a long definition.

Let us see some examples to clarify what is going on. Examples; consider the set $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Consider this set. Is this set bounded above? Yes, this set is bounded above. What is a good upper bound for this set? 1 is an upper bound. Note, I have not told you where this set is. So, let me just take it as a subset of $\mathbb{Q}$. In fact, 1 will be the least upper bound of this set. Why is that the case?

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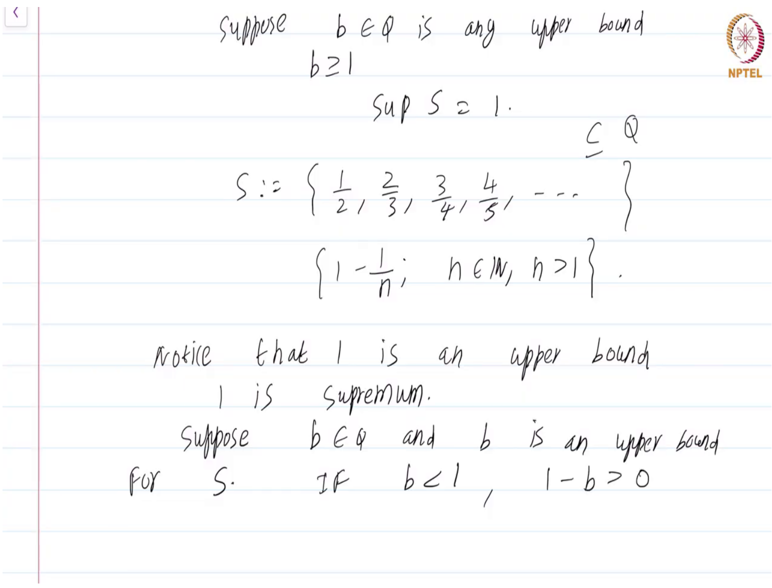

Well, suppose $b \in \mathbb{Q}$ is any upper bound, $b \geq 1$ simply because 1 is present in this set. For this reason, the property, the second property, if b is any upper bound for S, then $b \geq u$ is trivially satisfied. So, supremum, if you call this set S, $\sup S = 1$.

Let us see another example. It is a slight variant of this example. I define S to be, instead of $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$so on, I will define it as $S = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ..\}$. This is just $\{1 - \frac{1}{n} : n \in \mathbb{N}, n > 1\}$, ok. It is a collection of all numbers. Again, I will take this as a subset of $\mathbb{Q}$. It is a collection of all numbers $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ so on. I mean essentially $1 - \frac{1}{n}$.

Now, what is an upper bound for this set and does it really have a least upper bound? Well, let us see that. Notice that 1 is an upper bound, that is clear because all elements are of the form $1 - \frac{1}{n}$. Claim is that 1 is actually the supremum; 1 is actually the supremum.

Now, let us try to argue as follows. Suppose $b \in \mathbb{Q}$ and b is an upper bound for the set S, ok. If $b < 1$, we need to reach a contradiction, if $b < 1$, then $1 - b > 0$.

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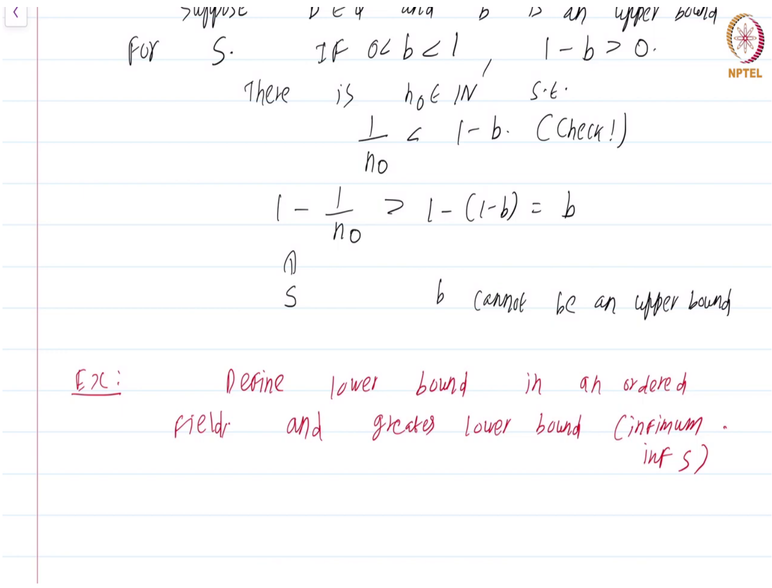

In fact, we can assume that this $b > 0$; because b has got to be an upper bound. So, you have $0 < b < 1$ and $1 - b > 0$, we have to reach a contradiction somehow.

Now, I will use one fact that I leave it to you to check. There is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < 1 - b$, ok. This is in fact known as the Archimedean property of the rational numbers. It is a consequence of the Archimedean property. We will see that in a few modules. However, to prove this is not at all hard for the rational number, so I urge you to try it right now.

So, check that you can find $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < 1 - b$, ok. Therefore, $1 - \frac{1}{n_0} > 1 - (1 - b) = b$. So, we have an element, this is an element in the set S that is greater than b. So, b cannot be an upper bound; b cannot be an upper bound. So, this proves that 1 is the least upper bound of the set S.

So, that is enough for the examples right now. Before we proceed with some more theorems I want to first make some remarks about infimum, rather I will just put it as an exercise for you to think about. Define lower bound in an ordered field and greatest lower bound. This is also called the infimum and that is denoted $\inf S$.

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$$1 - \frac{1}{n_0} > 1 - (1-b) = b$$

$\in S$ $\qquad$ b cannot be an upper bound

Ex: Define lower bound in an ordered field and greatest lower bound (infimum . inf S)

Show that an ordered field is complete iff any set that is bounded below has a greatest lower bound.

So, define lower bound and greatest lower bound in an ordered field. Show that an ordered field is complete if and only if any set that is bounded below has a greatest lower bound. So, do this exercise, you will have to define what lower bound is, you have to define what bounded below is, you will have to define what infimum is. You show that completeness is the same. You can reformulate completeness in terms of lower bounds instead of upper bounds.

Now, the definition of completeness seems very hard. It says that a field is complete if every set that is bounded above as a supremum or every set that is bounded below as an infimum. The reason is that any definition in mathematics that was arrived at after many years of work has this tendency to be a bit opaque because the distance between what motivated the impetus for studying a proper for figuring out a proper definition and the final definition coming about is many many years, because of that it is not really clear what is happening.

So, what we will do is we will now prove two more intuitive properties that complete ordered fields will possess called the Archimedean property and the nested intervals property. These two properties are more intuitive from our understanding of what a straight lines behavior is supposed to be. Before we get to these two results, let us first see a useful characterization of the supremum that will be of immense benefit to us in these proofs.

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iff any set that is bounded below has a greatest lower bound.

Proposition (Characterization of the supremum). Let F be an ordered field, $S \subseteq F$ bounded above. Let $u = \sup S$. Then for each fixed positive $\varepsilon \in F$, we can find an element $s \in S$ such that

$$u - \varepsilon < s \leq u. \quad (*)$$

Conversely, assume that u is an upper bound for S that satisfies $(*)$ then u is $\sup S$.

Proposition, this is characterization of the supremum. This result says the following. Let F be an ordered field ordered field, $S \subset F$ bounded above, let $u = \sup S$. Then for each $\varepsilon > 0$, $\varepsilon \in F$, we can find an element $s \in S$ such that $u - \varepsilon < s \leq u - - - (*)$.

Conversely, assume that u is an upper bound for S that satisfies $(*)$, then $u = \sup S$. So, this gives a complete characterization of the supremum. What this proposition says is the following. You have a set S in an ordered field that is bounded above. The supremum also exists which we are denoting by u. What it says is if you knock out a small portion of the supremum that is essentially what subtracting by a positive ε means then that ceases to be an upper bound of the set. There will be some element s with $u \geq s > u - \varepsilon$.

Needless to say, this choice of this element s depends on the choice of ε. If you choose a much smaller ε you will have to modify this element s in all probability ok.

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sup S.

Proof: Assume that for some choice of $\varepsilon > 0$, $(*)$ is false. Then

$\underbrace{u - \varepsilon \geq s}_{\text{upper bound.}} \quad \forall s \in S.$ (Check that this is the correct negation)

$u - \varepsilon < u$ which contradicts property (ii) of the supremum.

Suppose S is bounded above and u is an upper bound that satisfies $(*)$. Then we have to show $u = \sup S$

Proof, the proof is a proof by contradiction. We had already seen one or two proofs by contradiction, specifically the irrationality of $\sqrt{2}$. The essential idea is that we will assume that what we are trying to prove is false and add it to the list of hypotheses that we have. Then we can play around with the given hypothesis along with our new assumption and reach a contradiction, then by the law of excluded middle our assumption must be false.

This technique is really powerful especially when we do not know how to start a proof. You have a theorem you do not know how to prove it. Just add the negation of what you want to prove as a hypothesis, as an assumption and proceed.

So, first we will prove the first part not the converse. Assume that for some choice of $\varepsilon > 0$, $(*)$ is false. Then, we have $u - \varepsilon \geq s$ for all $s \in S$, right?. Our $(*)$ says we can find some element s that satisfies the inequality, $u - \varepsilon < s \leq u$, the negation of this statement is that for some choice of $\varepsilon > 0$ and all choices of s, $u - \varepsilon \geq s$. Check that this is the correct negation.

But, $u - \varepsilon < u$, why? because ε is positive; because ε is positive $u - \varepsilon < u$, which contradicts property (ii) of the supremum. You cannot have an upper bound, $u - \varepsilon$ is going to be an upper bound, you cannot have an upper bound that is strictly smaller than the least upper bound.

So, this contradiction shows that the supremum will necessarily have property $(*)$. On the other hand, suppose set S is bounded above and u is an upper bound that satisfies $(*)$. Then we have to show $u = \sup S$, ok.

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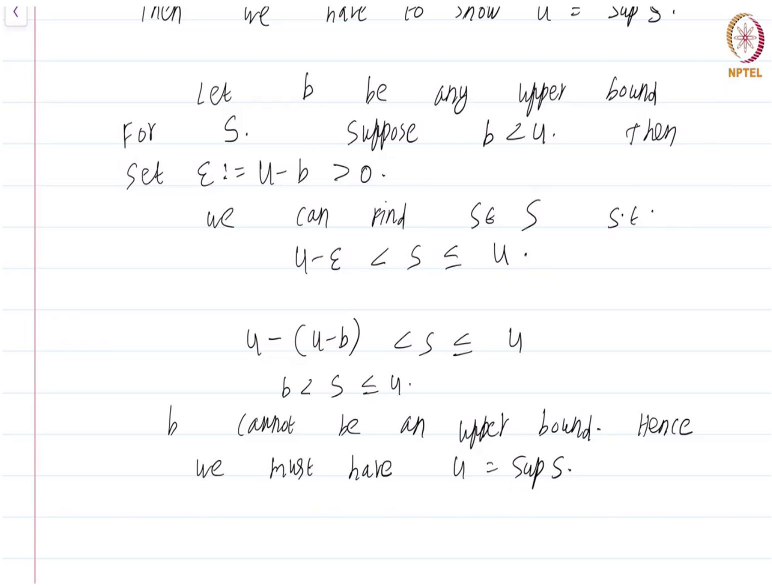

How does one show this? well, let b be any upper bound for the set S. Now, suppose $b < u$, then set $u - b = \varepsilon$,, set this to be ε. Then $\varepsilon > 0$ because $u > b$. Now, because of the property $(*)$, we are now assuming that $(*)$ is satisfied, we definitely have, we can find $s \in S$ such that $u - \varepsilon < s \leq u$. This is simply property $(*)$.

But, $u - \varepsilon = u - (u - b) < s \leq u$. In other words $b < s \leq u$, ok. So, b cannot be an upper bound; cannot be an upper bound. Hence we must have $u = \sup S$. So, this completes the proof.

Now, what we will do is we will proceed and prove the nested intervals theorem and the Archimedean property in the next module. This is the course on Real Analysis and you have just watched the module on Completeness.