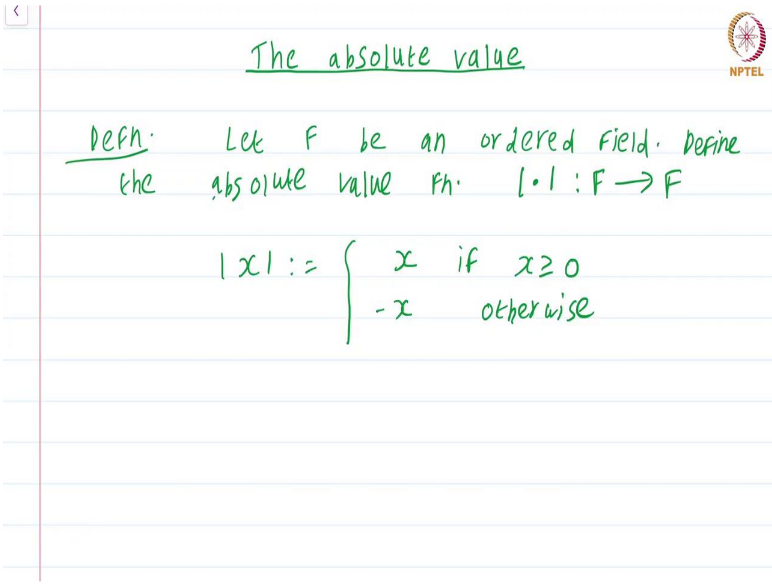
Real Analysis - I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 4.3
Absolute Value

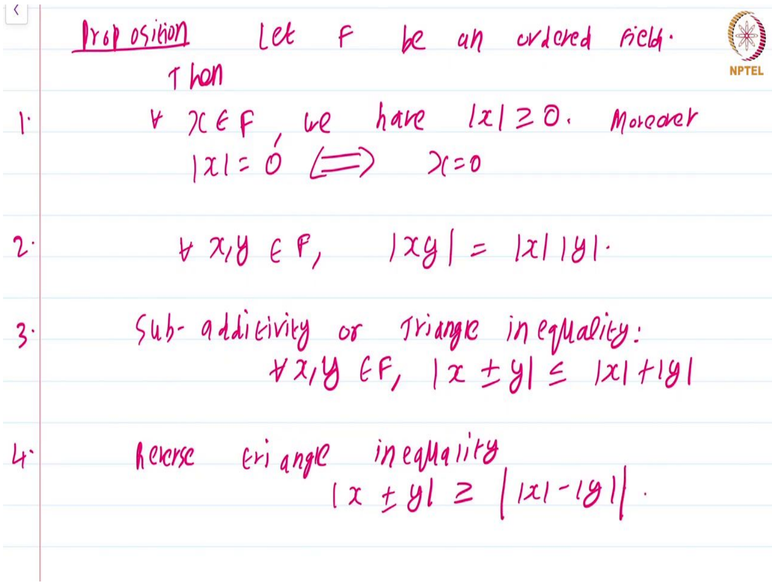
(Refer Slide Time: 00:13)



We begin with a definition. Let F be an ordered field. Define the absolute Value Function denoted $|\cdot|$. Note that I have put a bold dot that is supposed to denote, where the function is plugged in. This is sort of a weird notation for a function, but it is commonly used in mathematics.

The definition is $|x| := x$ if $x \geq 0$ and $-x$ otherwise. So, this definition is no doubt familiar to you from your studies before. The absolute value in the rational numbers $\mathbb{Q}$ has a nice geometric description, interpretation as the distance from the origin. This absolute value will always give you a non negative element. It is designed in such a way that if it is positive, it returns back the same number; if it is negative, it returns the negative. So, you will get always a positive answer to this. So, we will require only the basic properties of the absolute value. One big inequality we will require later in the series , it is better to do it right then and there.

(Refer Slide Time: 02:09)



So, let us state a theorem or rather a proposition, it is not such an important thing to call a theorem.

Proposition: let F be an ordered field. Then we have various properties,

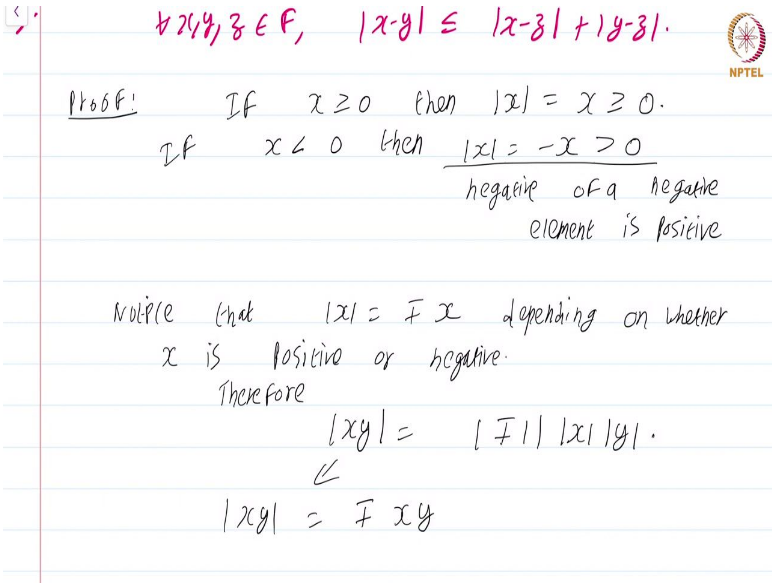
1. $\forall x \in F$, we have $|x| \geq 0$. Moreover, |x| = 0 if and only if x = 0.
2. $\forall x, y \in F,\ |xy| = |x||y|$.
3. This is the most important sub additivity or triangle inequality. So, the name triangle inequality comes from the fact that the sum of two sides of a triangle is at least as large as the third side. One needs to go to several variables to exactly see why this is called a triangle inequality . We will see that in the section on topology.

   $\forall x, y \in F\ |x \pm y| \leq |x| + |y|$. So, this is actually two inequalities that I have condensed into one inequality by using the $\pm$ sign.

4. Reverse triangle inequality, the reverse triangle inequality says the following. It says $|x \pm y| \geq ||x| - |y||$.

   Why it is called the reverse triangle inequality is easy to see. In fact, I can put a modulus sign here on the right hand side as well .

(Refer Slide Time: 04:32)



5. $\forall x, y, z \in F \ |x-y| \leq |x-z| + |y-z|$.

So, it is a simple list of properties almost all the properties can be given very simple proofs. Let us see the proof of this.

First is the obvious thing that $|x| \geq 0$. This is obvious, but nevertheless let me write down a proof. If $x \geq 0$, then $|x| = x \ \geq 0$. If x < 0 then $|x| = -x \ \geq 0$, where the fact that if x < 0, then -x > 0 follows from the fact that the negative of a negative element is positive; it is one of the properties of an ordered field.

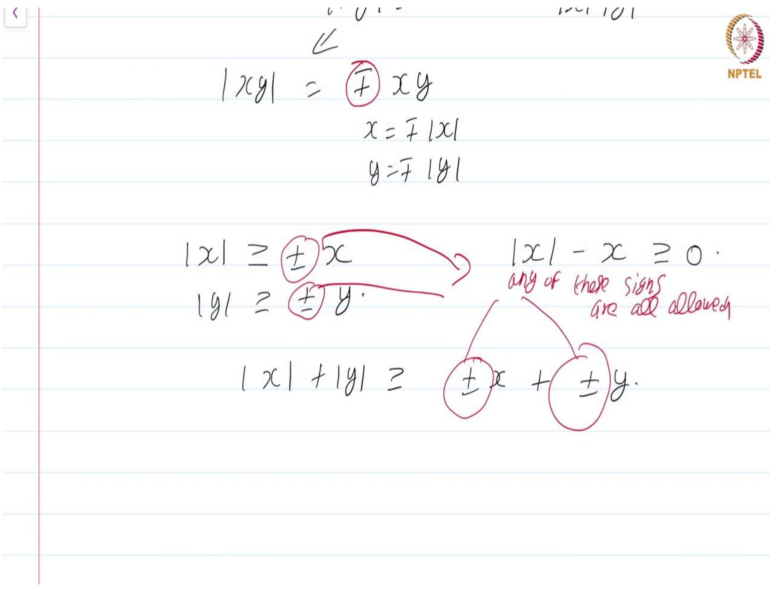
Now, let me make a remark here. Here the proof is what is known as a proof by cases. We want to prove something more precisely, we want to prove that $|x| \geq 0$ irrespective of what x is. What we do is, we split into two cases. The first case is when $x \geq 0$, the second case is when x < 0, then we proceed and give an independent proof for both cases. This will prove that whatever we are trying to assert is indeed true for all elements x coming from the field F.

So, proof by cases is probably one of the most common techniques used in a proof, especially in really long proofs, where you will not just have two cases you will have dozen or even several thousands in some of the deeper theorems.

Now let us try to prove the rest. First let us make an observation. Notice that mod x is always $\pm x$,depending on the sign of x depending on whether x is positive or negative.

So, the modulus or the absolute value is always going to be the same as what you are going to put as an input except possibly the sign could change well that it immediately gives. Therefore, |xy| is certainly going to be $|\pm 1||x||y|$. How did I get this? Well, it is got to be $|xy| = \pm xy$ and we already know that x is going to be $\pm|x|$. I am just reversing the role; $|x| = \pm x$.

(Refer Slide Time: 08:12)



Therefore x has to be $\pm|x|$. Similarly, y has to be $\pm|y|$. The net upshot of all this is |xy| has to be $\pm|x||y|$ and we know that this sign also you need to put a modulus there simply because |xy| will always be non negative. So, $|\pm 1|$ can be immediately gotten rid of. So, it immediately follows that |xy| = |x||y|.

Now, let us deal with the sub additivity or triangle inequality. Well from all these remarks, it should be clear to you that |x| is going to be greater than or equal to $\pm x$ always. Why does this follow? This follows because observe that |x| - x will always be greater than or equal to 0. Check why this is true.

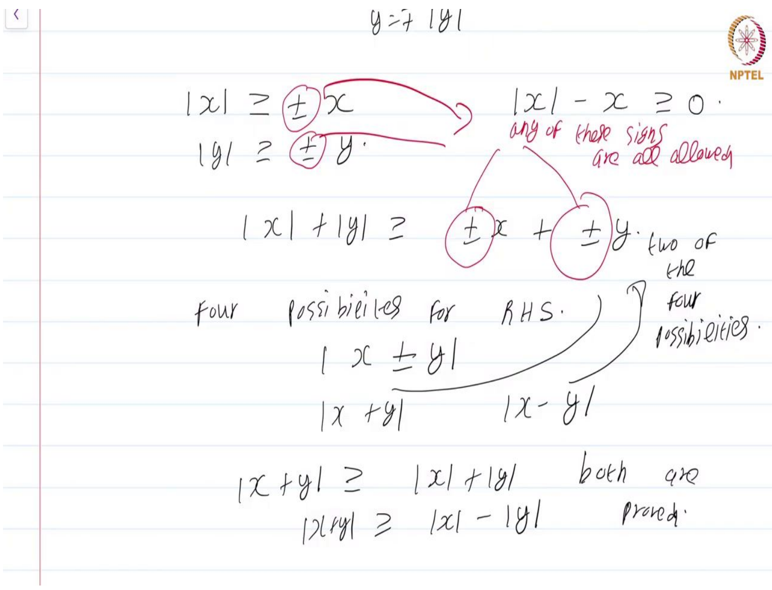
So, $|x| \geq \pm x$. Similarly, mod $|y| \geq \pm y$. Now, we immediately get by one of the properties of an ordered field that $|x| + |y| \geq \pm|x| + \pm|y|$. Actually to be 100 percent precise this is a bit confusing. I will just write $|x| + |y| \geq \pm|x| + \pm|y|$.

Now, I am being very sloppy in the notation because the proofs here are quite easy, but I should not be too loose so as to confuse you. First notice that when I wrote $|xy| = \pm xy$ here or $|x| = \pm x$. I do not mean both possibilities. I just mean that |x| is going to be either + x or -x depending on the sign of x ditto for $|xy| = \pm xy$.

But here I am being a bit loose to simplify on the amount of writing. For brevity sake, I am writing $|x| + |y| \geq \pm x + \pm y$. Here, I mean irrespective of what combination of the $\pm$ signs you have for x and y this inequalities still true. So, in these three places these three places or rather four places you should take it as any of these signs are allowed any of these signs are allowed.

In the previous cases only one of them, the inequality is true for only one of them; |xy| = xy or it is equal to -xy, it is not equal to both. Now you have $|x| + |y| \geq \pm x + \pm y$, where all possibilities are allowed.

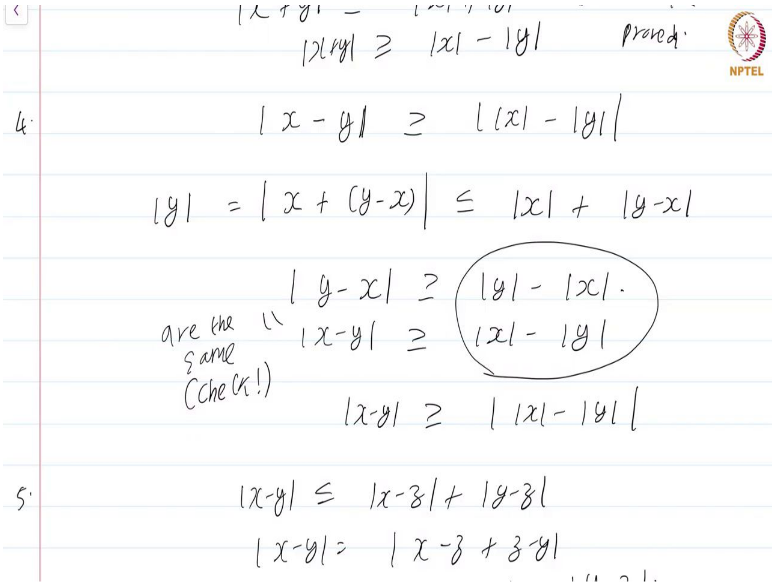
(Refer Slide Time: 11:49)



There are four possibilities for the right hand side for the right hand side. Now, observe that these quantities $|x \pm y|$. We want to show that $|x + y| \geq |x| - |y|$. Observe that these quantities |x+y| and |x-y|, these are two of the four possibilities here.

So, check that two of the four possibilities here are indeed going to be |x+y| and modulus of |x - y|. That means, we immediately get $|x+y| \geq |x| + |y|$ and $|x+y| \geq |x| - |y|$, we get both. So, that completes the proof of sub additivity or triangle inequality.

(Refer Slide Time: 13:29)



Now, how do you prove the reverse triangle inequality? That is not too hard either. We want to show let us me just show one part. Let me just show you that |x+y| is or rather let me show the minus sign that seems that looks like a bit harder; $|x-y| \geq ||x| - |y||$.

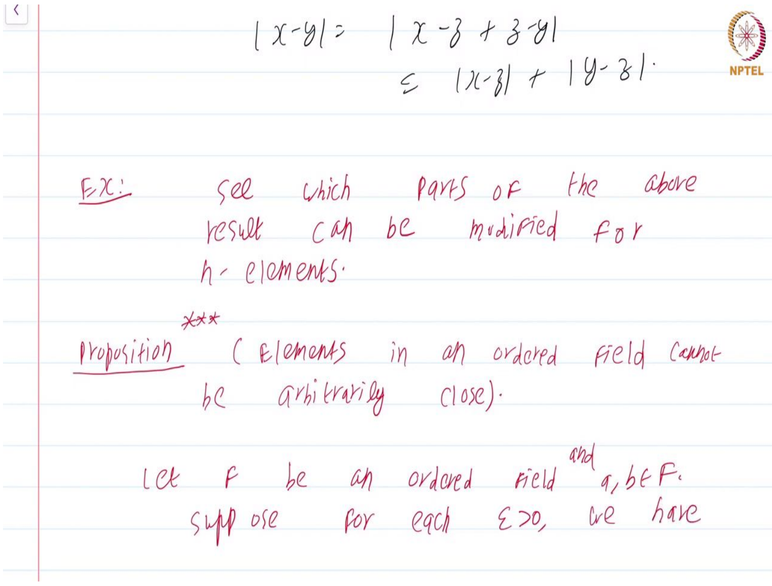
Now, how do you show this? Well, look at |x + (y-x)|; this is by the triangle inequality < |x| + |y-x|, but this is nothing but |y|. So, you immediately get $|y-x| \geq |y| - |x|$.

So, we also get |x-y| is by the same argument is $\geq |x| - |y|$, but |y - x| and |x - y| are the same. Again check all things that are trivial, which can be checked in a line or two. I am leaving to you. So, putting these two together we get $|x-y| \geq ||x| - |y||$.

Why did we get these because ||x| - |y|| has to be one of these two . That is the reason. So, that proves the reverse triangle inequality.

Of course, I am leaving another case, but that is equally simple. Finally, we want to show that for all x, y, z $|x-y| \leq |x-z| + |y-z|$. This immediately follows by writing |x-y| has mod |x-z + z-y| and then this will immediately give |x-z| + |y - z| by triangle inequality and we are done.

(Refer Slide Time: 16:15)



So, this concludes the proof of these basic properties.

Now, let me give you an exercise.

See which parts of the above result can be modified for n elements? There are only two elements or at the max three elements. Think over which of these properties can be extended to many elements, ok.

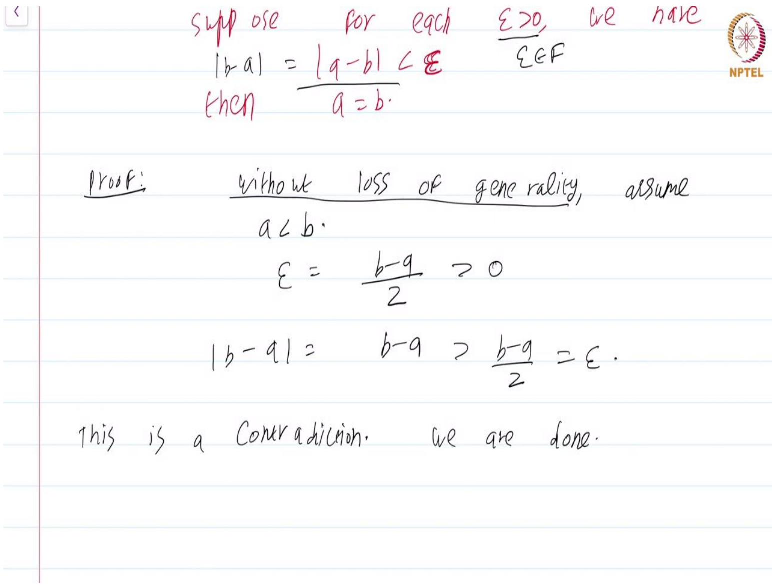
Now I am going to end this section with a simple proposition which is so simple that I will never ever explicitly mention that this proposition is being used in the course.

So, let me write down this proposition and let me put a double star or a triple star next to the proposition. This is to indicate that this proposition has to be committed to your memory. If I wake you up in the middle of the night and ask you what is proposition about elements in an ordered field being arbitrarily close you have to know it. So, some amount of memorization is impossible without any I mean some amount of memorization is unavoidable in any subject of studies and this is one of them.

Elements in an ordered field cannot be arbitrarily arbitrarily close, what this proposition says is rather simple.

Let F be an ordered field $a, b \in F$. Suppose for each $\epsilon > 0$, we have modulus of $|a - b| < \epsilon$ then a = b.

(Refer Slide Time: 18:34)



What is this proposition trying to say? It is saying that regardless of what value of ϵ you choose , being any positive element in the field. Note this ϵ is coming from F. Let me be precise. ϵ is coming from F, irrespective of what epsilon you choose to be positive coming from the field. If it so happens that |a - b| is less than that then a is force to be equal to b.

Why is this the case? Well, first let me give a proof trip. We are going to apply a proof, but prove only one case. Remember when I made a comment about proof by cases, where we split the number of cases into two or three and prove each case independently. Here what we will do is we will say without loss of generality without loss of generality assume a < b.

What does this mean? Essentially, I am going to appeal to a proof by cases. But if I prove the case a < b then there is no necessity for me to prove the case b < a. Why is that because at the end of the day, a and b are just symbols. If I can show that when a is less than b the result is true; obviously, the same has got to be true and b < a because the final result does not give any priority to the role of a on to the role of b.

The way it is written you have a feeling that a and b indeed have asymmetric rules. But that is not the case, because a - b is same as b - a, modulus of course, |a-b| is same as modulus of |b-a|. So, there is no distinguished role for a or b. So, if I can prove the case a < b then I am done for the case if b < a also.

So, I am going to assume that a < b and reach a contradiction. That means, if I assume b < a, I will also get a contradiction, which leaves the only possibility that a has got to be equal to b . So, then in such situations when you are making a proof by cases, but the proof by one case is actually enough to deal with all other cases we use this term without loss of generality. That means, even though you are making an additional assumption, this additional assumption does not impact the validity of the proof for all the cases at once.

So, how does this go? Well, I want to show that this leads to a contradiction. The fact that is given to me is that |a-b| can be made less than ϵ, no matter what ϵ that is. So, I have to choose a special epsilon. I choose ϵ to be $\frac{b-a}{2}$. Now this is be a positive quantity simply because a < b. Now, |b-a| = b-a because b > a, which is greater than $\frac{b-a}{2} = \epsilon$ which is a contradiction, this is a contradiction.

So, since this case leads to a contradiction, so, will the other case and we are done. So, two elements in an ordered field cannot get closer and closer to each other. Recall that we had interpreted the modulus as the distance to the origin in the case of rational numbers. So, in a similar vein you can think of |a-b| as the distance between a and b, this just says that the distance between a and b cannot get closer and closer .

This is a course on real analysis and you have just watched the lecture on the absolute value.