Real Analysis - I
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Lecture - 3.4
Cardinality

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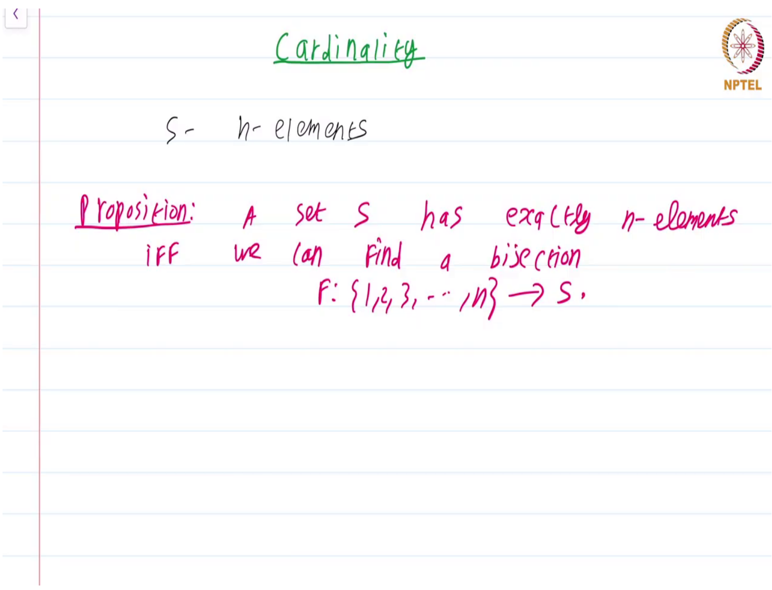

Suppose, you have a set that has n elements, let us say set S that has n elements. How will you demonstrate that it indeed has exactly n elements? It might be helpful to consider a real world situation, say, somebody hands you over a set of 500 rupee notes and says that there are 5000 rupees in net in these notes together. How do you check that? Well, you start rifling through the notes one by one saying 1 2 3 4 and the number you assign to the very last note is the number of notes that are there.

In short, what we are doing is to each note we are assigning a unique number starting from 1 and the ending number is supposed to be the number of elements. From this real world analogy the following proposition is very very easy to prove rigorously and I leave it to you to prove that. The proposition says the following.

A set S has exactly n elements if and only if we can find a bijection $F : \{1, 2, 3, ..., n\} \longrightarrow S$, the set consisting of the first n natural numbers to the set S.

This is analogous to just pointing to the first element and saying 1, pointing to the second element and saying 2 and pointing to the very last element and saying n, ok.

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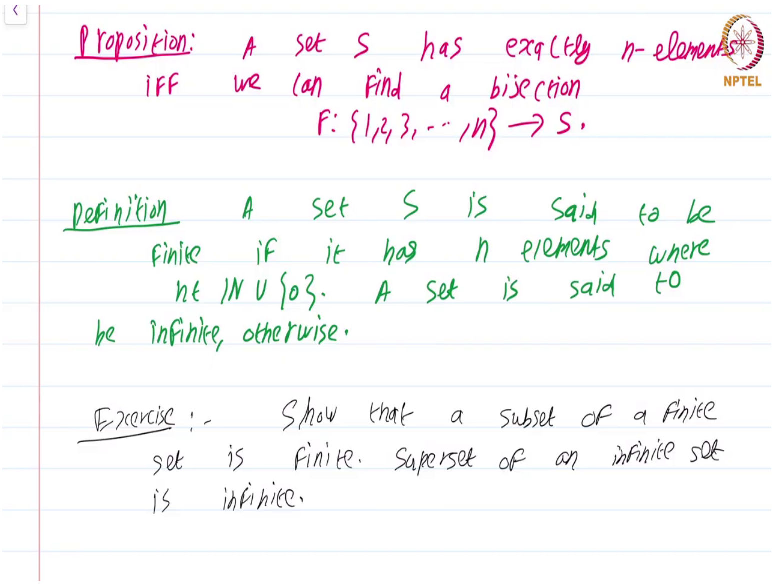

So, we have this following definition that comes.

Definition: A set S is said to be finite; to be finite if it has n elements where $n \in \mathbb{N} \cup \{0\}$. A set is said to be infinite otherwise.

So, finite sets are those that have n elements for some natural number and 0. So, the empty set which has 0 elements is considered a finite set and an infinite set is one that is not finite.

Now, we have this easy exercise. Show that a subset of a finite set is finite, also show that superset of an infinite set is infinite. This exercise is very straightforward. Please solve it.

The next proposition that I am about to prove is also utterly obvious, but there is an idea in the proof. So, I am going to present the proposition.

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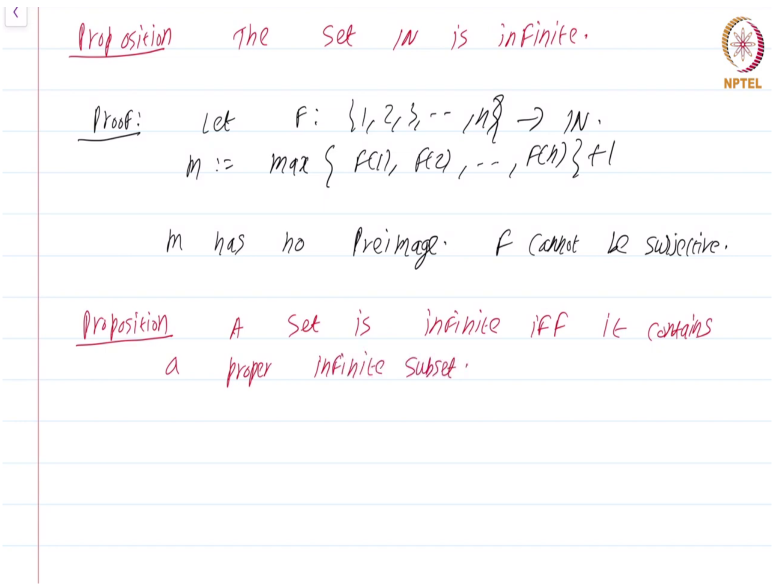

Proposition: The set $\mathbb{N}$ is infinite.

The proof is very easy, but there is a worthwhile idea. to remember. We have to show that you cannot find a bijection from any set of n elements to the natural numbers. What you do is, you consider a map $F : \{1, 2, ..., n\} \longrightarrow \mathbb{N}$.

Then set $m := max\{F(1), F(2), ...F(n)\} + 1$, then it is clear that m has no pre-image. That means F cannot be surjective, ca. What we have shown is any mapping from an n element set to the natural numbers cannot be surjective, therefore N is infinite.

The next proposition is also not that hard, but it is a little counterintuitive.

Proposition: A set is infinite if and only if it contains a proper infinite subset..

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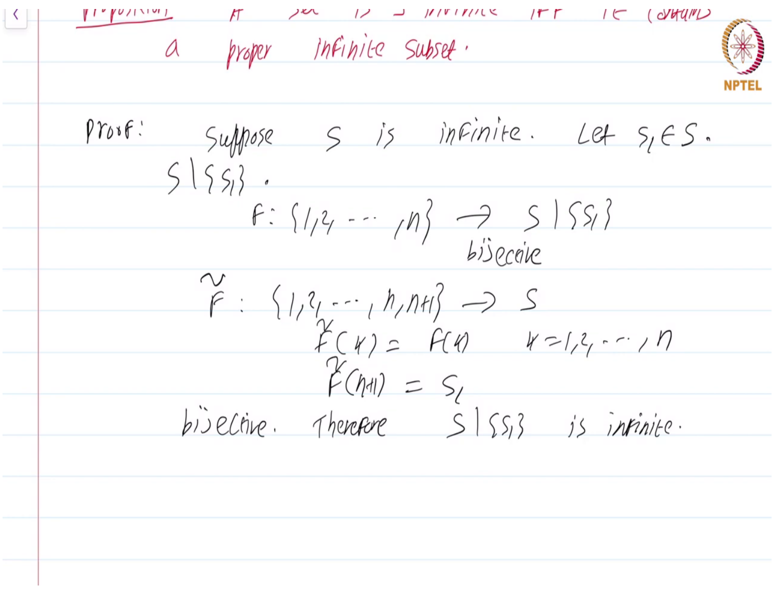

How do you prove this? Well, it is an if and only if statement. So, there are two directions to prove. The fact that. if a set contains a proper infinite subset, then the set itself has got to be infinite just follow from the last exercise that I have given you that the superset of an infinite set is always infinite, ok. So, now I will prove the other direction.

I will assume, suppose S is infinite. I have to produce for you a proper infinite subset. How do I do that? Well, it is very simple.

Let $s_1 \in S$, consider $S \setminus \{s_1\}$. Removing one element from an infinite set is suddenly not going to make it finite. How do I prove that? Well suppose you have a map $F : \{1, 2, ..., n\} \longrightarrow S$,that is bijective.

Suppose, you are able to find a map like this, then the map F extended to let me just call it $\tilde{F}$: $\{1, 2, ..., n+1\} \longrightarrow S$, where it is defined as

$\tilde{F}(k) = f(k), \quad k = 1, 2, .., n$ and $\tilde{F}(n+1) = s_1$. This map is clearly bijective.

If you could find a map , then you can find a bijective map t $\tilde{F}$: $\{1, 2, ..., n+1\} \longrightarrow S$, which is not possible. Therefore, $S \setminus \{s_1\}$ is infinite as required. So, infinite sets are those that have an infinite proper subset. I will give you another exercise that immediately follows from our discussions.

Exercise: Show $\mathbb{Z}$ and $\mathbb{Q}$ are infinite.

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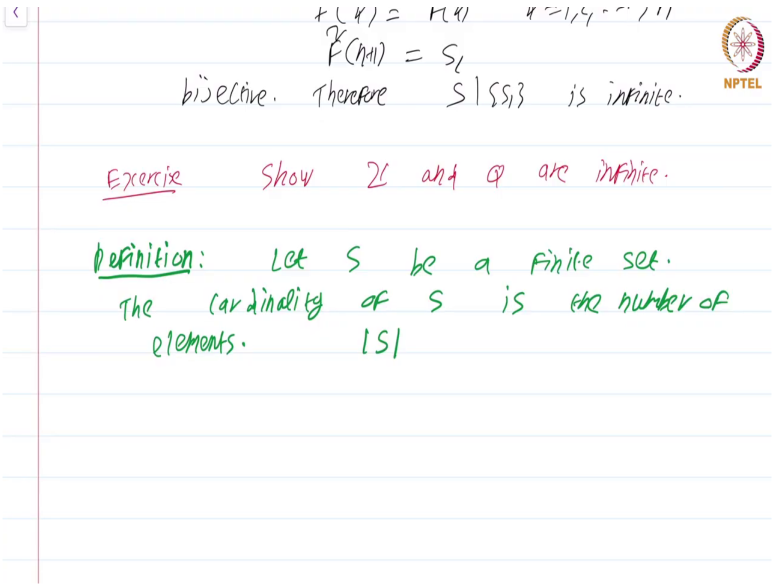

This just follows immediately from our discussion. In fact, it follows from the previous exercise quite easily. So, for finite sets I can define a notion called cardinality that is as follows.

Definition: Let S be a finite set. The cardinality of S is the number of elements. Usually we denote cardinality by | |. We put a modulus sign of sorts.

Now the question arises, can we define a cardinality for infinite sets, we can, but things are a bit tricky for infinite sets. Let me first, before we discuss this in greater detail, prove that any infinite set is strictly larger, not strictly larger sorry is at least as large as the natural numbers. How do we make that precise?

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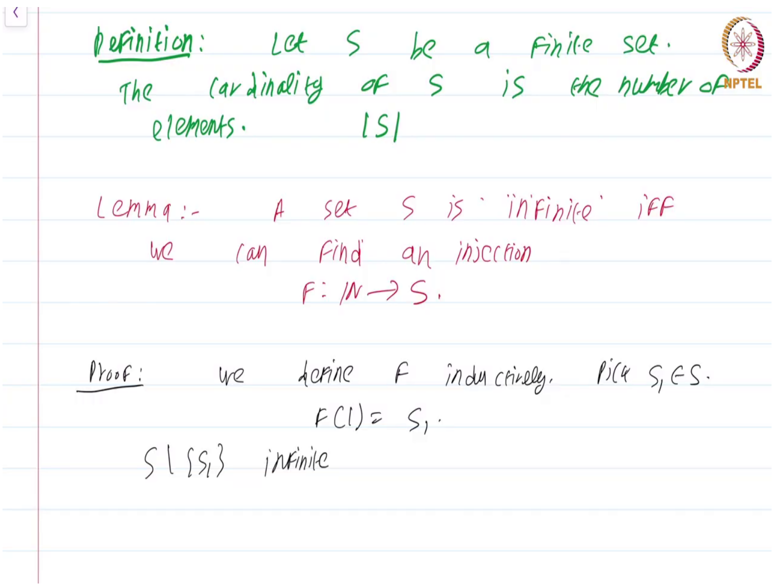

We have the following Lemma. A set S is infinite if and only if we can find an injection $F : \mathbb{N} \longrightarrow S$.

Now one part I am going to leave it to you to show that if indeed you can find an injection from the natural numbers to the set S, then the set S has to be infinite. That part I leave to you.

Now I am going to show that an infinite set always admits an injection from the natural numbers. How do we show that? We define F inductively, we define F inductively. How is this done? Well you first pick an element $s_1 \in S$. It could be any element, it doesn't really matter.

Set $F(1) = s_1$. Now the set $S \setminus \{s_1\}$ is infinite, that is just what we showed in the last proposition.

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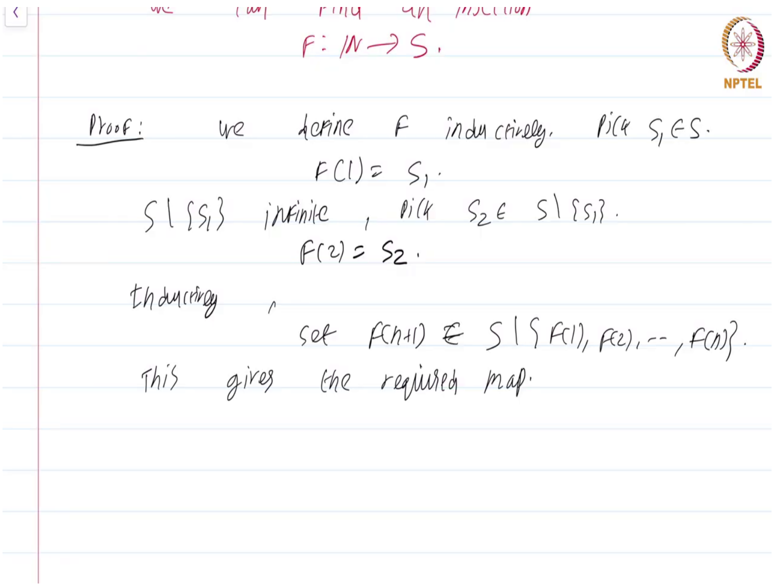

So, you can pick $s_2 \in S \setminus \{s_1\}$. We can pick an element s_2 from the set S with s_1 removed, then set $F(2) = s_2$. So, inductively suppose you have chosen $F(1), F(2), \ldots, F(n)$, set $F(n+1) \in S \setminus \{F(1), F(2), \ldots., F(n)\}$, pick any element ok. This gives the required injection, this gives the required injection.

So, inductively define the map to ensure that each stage it is injective. Therefore, globally also it is injective, required map.

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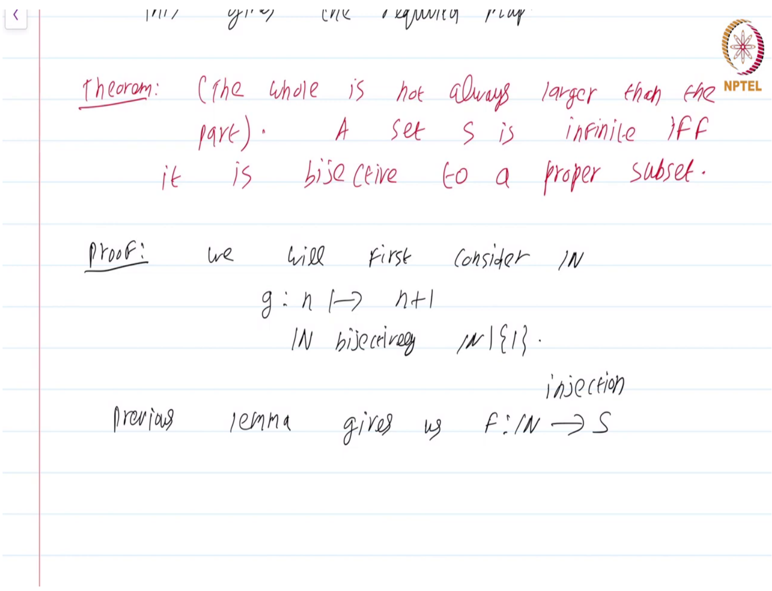

Now, we come to a mind bending theorem which is an important theorem. I like to call this theorem the whole is not always not always larger than the part. What does this theorem say? A set S is infinite if and only if it is bijective to a proper subset, ok.

So, the theorem says that infinite sets are those that are in themselves bijective to a proper subset. Now, we have already seen that if you remove a single element from an infinite set, you still get an infinite set. This theorem sort of extends that to a much more powerful result proof.

Before I present the proof let us just remark what we have to do in this set. In this particular theorem we are given a set. We have no further data about that set other than the fact that it is infinite, from that we have to manufacture a subset and a bijection. This seems like a hard task. So, what we do in such situations and it is generally a good idea to do this is to first concentrate on a particular case, fix S to be a particular set which is infinite and try to prove it in that case. In this particular theorem what happens is that particular case itself gives you the full proof, but usually that does not happen, but at that very least what happens is when you show a particular case, you get an idea of how to treat the general result.

So, what we will do is we will first consider, first consider $\mathbb{N}$ the natural numbers, ok. Now, we have an obvious bisection g which takes $n \mapsto n + 1$. This maps $\mathbb{N}$ bijectively to $\mathbb{N} \setminus \{1\}$.

So, we have managed to solve this theorem in the special case when the set S, the natural numbers. How do we deal with the general case? The previous lemma says; gives us $F : \mathbb{N} \longrightarrow S$, an injection. We get an injection from the natural numbers $\mathbb{N}$ to the set S.

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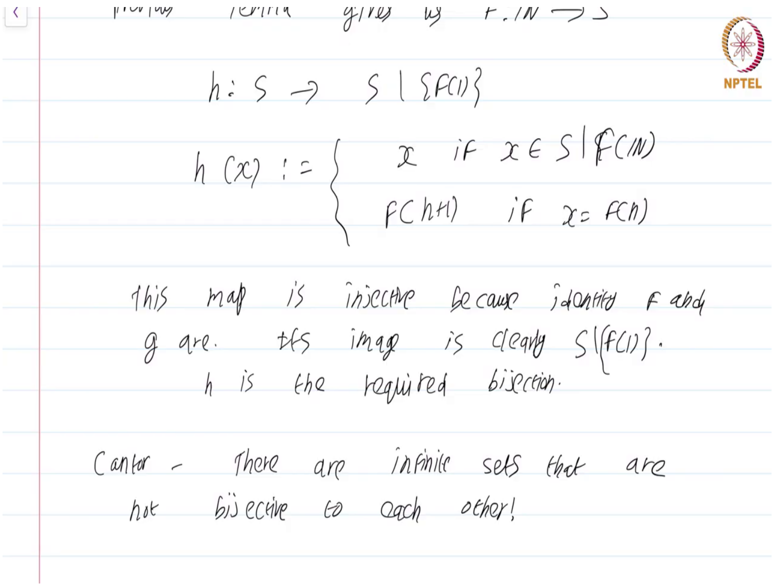

Now what I do is, I define $h : S \longrightarrow S \setminus \{f(1)\}$. The image of 1, how do I define this? I define this in the following way, $h(x) := x \ if \ x \in S \setminus f(\mathbb{N})$. So, if you take a particular x which is not in the image of the map F, just fix it just take the identity there. Otherwise just set it to be $f(n+1) \ \ if \ x = f(n)$. So, what we have done is essentially we have defined the map h in two parts. On the set that is not hit by the image of F you just leave it as it is, on the part that is hit by the image of F, you just move it to the side using the map g.

So, this map is injective because the identity, f and g are ok, its image is clearly $S \setminus \{f(1)\}$. Every element other than the element f(1) is considered, is there in the image of this map, ok. So, h is the required bijection, h is the required bijection.

So, this concludes the proof. So, infinite sets or those that are bijective to a proper subset ok. So, this sort of says that defining cardinality for infinite sets could be a bit tricky. Now one guess would be that all infinite sets are bijective to each other, this particular theorem might suggest that this is not true.

This was shown by Cantor. Cantor showed that there are infinite sets that are not bijective; that are not bijective to each other. Not only did he show that, he showed that there are hierarchies, there are infinitely many different infinite sets.

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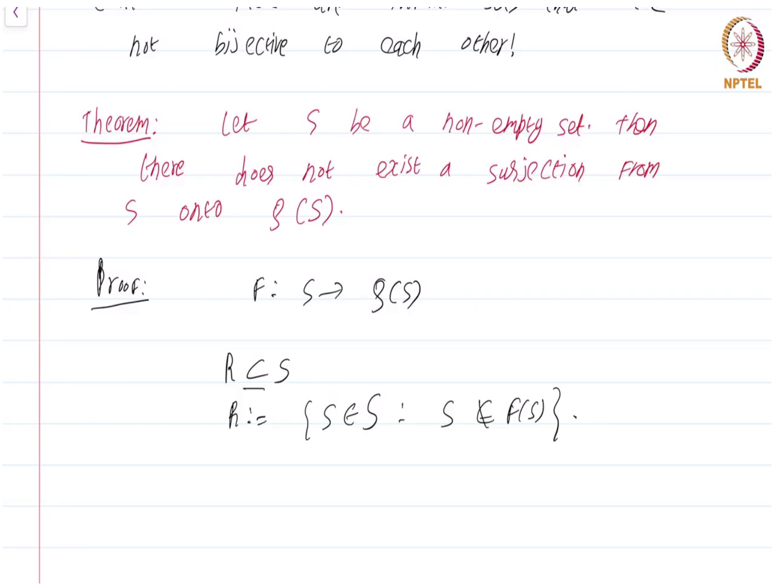

And that is the content of the next theorem. This is proved by Cantor. The theorem is as follows.

Let S be a non-empty set, let S be a non-empty set, then there does not exist a surjection from S onto the power set P(S).

How do you prove this? The proof is an ingenious idea of Cantor. What you do is suppose you have a map $f : S \longrightarrow P(S)$.

Now what you do is, you define a special subset $R \subset S$ defined as follows.

$R := \{s \in S : s \notin f(s)\}$. You look at all those elements in the set S that are not present in the image. That means, the image does not possess the element s . Suppose this map is surjective.

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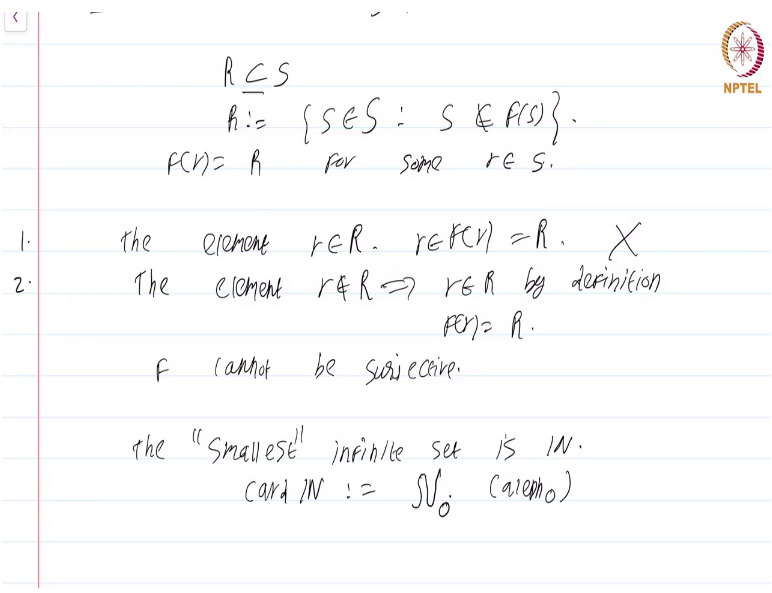

Then we must have some $f(r) = R$ for some $r \in S$. Now two possibilities can happen. The first possibility is that the $r \in R$ that can happen. There are only two possibilities, either the element $r \in R$ or it is not.

If the element $r \in R$, then this is problematic because $f(r) = R$ but the set R was defined precisely to be those elements which are not contained in its image. So, this possibility is ruled out. This is not possible.

The second possibility is the element $r \notin R$, but this means $r \in R$ by definition because $f(r) = R$. So, the definition of the set R is it is those elements that are not contained in its image and this is directly contradicting that, ok.

So, both possibilities give a contradiction. That means F cannot be surjective, F cannot be surjective. So, the power set of a non-empty set is always larger than that set. The smallest infinite set, the "smallest" infinite set is $\mathbb{N}$. The smaller infinite set is the natural numbers, the cardinality of the natural numbers is defined to be this symbol called $\aleph_0$.

Now, to define what this is precisely will require a much deeper set theoretic machinery, then what we have available, I am just mentioning this in the passing, this is read aleph naught, aleph 0 and I will try to write it correctly. It is a bit complicated. Write it as, it is Hebrew, I think. It is supposed to be written something like this.

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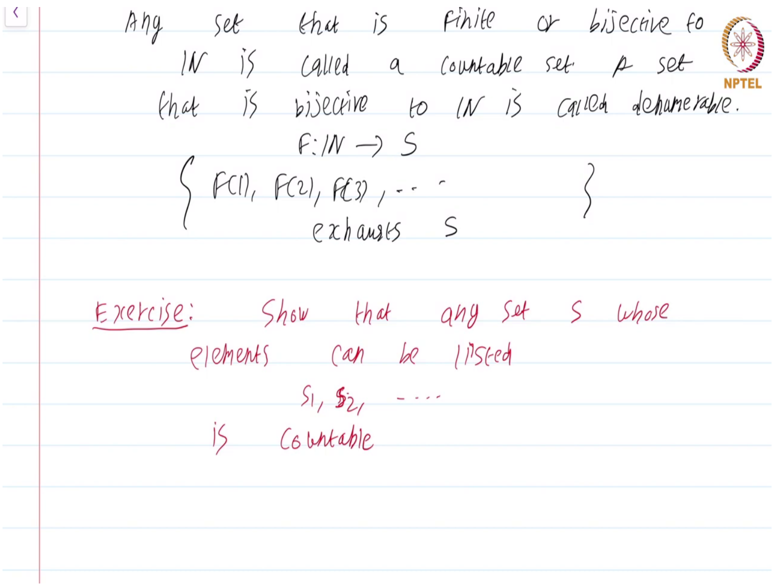

Now one more definition; any set that is finite or bijective to $\mathbb{N}$ is called a countable set. A set that is bijective to $\mathbb{N}$ is called denumerable . Now denumerable sets by definition admit a bijection from the natural numbers to itself ok.

So, we can write, you can write the elements of S as $F(1), F(2), \ldots..$ This collection $\{F(1), F(2), F(3), \ldots..\}$ will exhaust S. Why does it exhaust S? Because that is the definition; it is a bijection from natural numbers to S. Now I give you an exercise.

Show that any set S whose elements can be listed $s_1, s_2, \ldots.$ is countable. Note that I am not requiring this list to have no repetitions. In the previous list when the set is denumerable, when I write the set as $\{F_1, F_2, F_3, \ldots.\}$ .There is no repetition in the list, but here in this exercise I am not requiring the list not to have repetitions. It can have repetitions, but the conclusion is weaker. The conclusion is that the set is countable. It could be denumerable, it could be finite, it really does not matter ok.

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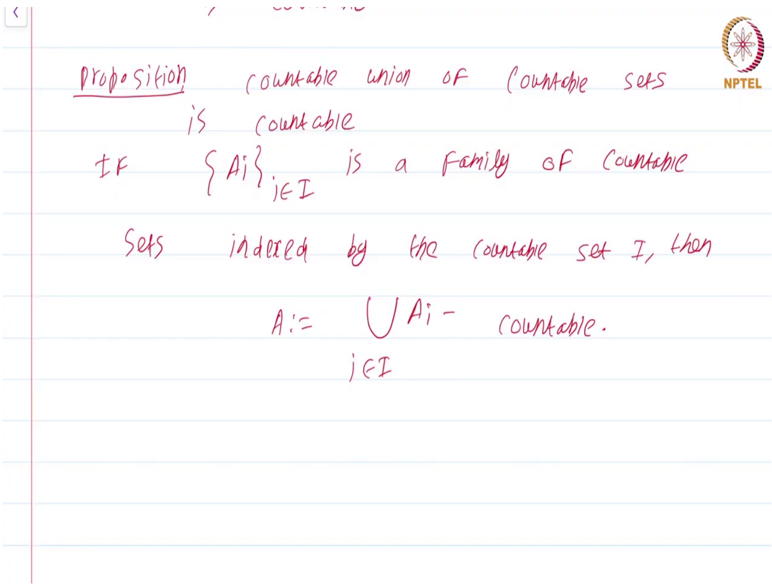

With this exercise in hand, we can now end this module on a cardinality with a nice proposition.

Proposition: Countable union of countable sets is countable .

How do I make this precise? Well if you consider the collection of sets $\{A_i\}$, where I is an indexing set, is a family of countable sets indexed by the countable set I , then the set $A = \bigcup_{i\in I} A_i$, this is also countable.

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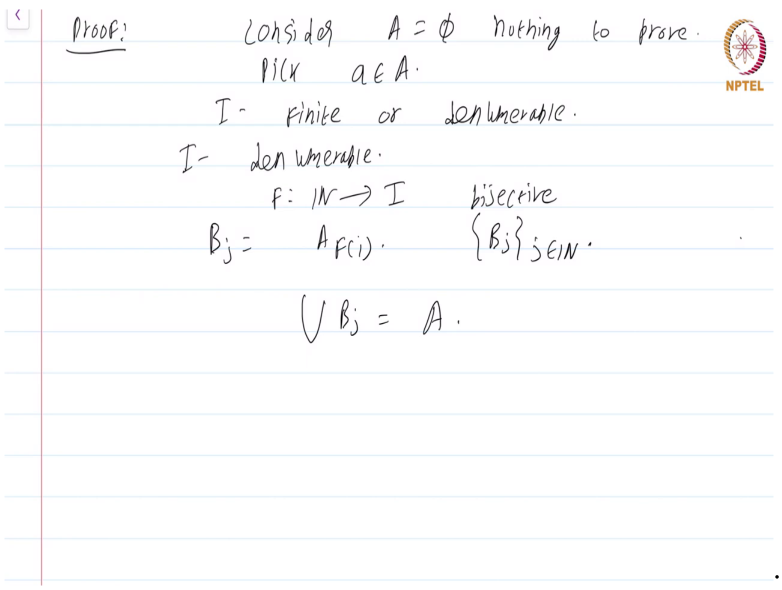

Proof. First consider $A = \phi$. In this scenario nothing to prove. So, we have disposed off with the trivial case that A is the empty set.

So, pick an element $a \in A$. Now what we do is the following, there are two possibilities that arise. Either I could be finite or denumerable.

First let me consider the case where I is denumerable. I will consider the case I is denumerable and what I will do is, I take a map $F : \mathbb{N} \longrightarrow I$ that is bijective and define $B_j = A_{F(i)}$.

So, these B_j are a collection of sets indexed by the $\mathbb{N}$ and not only that $\bigcup B_j = A$, clearly. I have just reindexed the set A with an index coming from the natural numbers.

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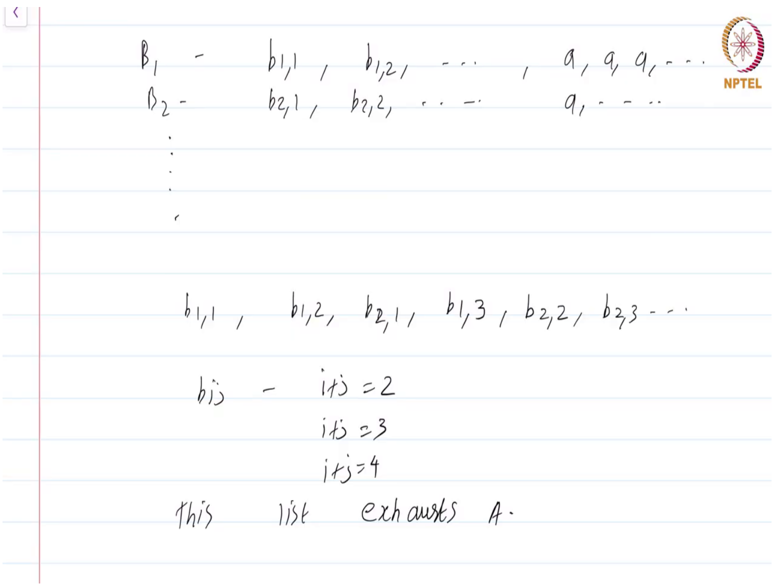

Now what I do is, I look at B_1 and list out all the elements of B_1 which is possible because B_1 is countable $b_{1,1}, b_{1,2}, \cdots\cdots$so on. But this list could terminate. If the list happens to terminate what I do is, I extend the list infinitely by just adding a's I just keep repeating a's in this list. I do the same thing for B_2, I write it as $b_{2,1}, b_{2,2}, \cdots$ and so on.

If the list does happen to terminate, I just repeat the a's. So, I get an infinite rectangular array of elements coming from the various B_i's. Now I am going to list all the elements of a as follows. First I will put $b_{1,1}$, then I will put $b_{1,2}$, then I will put $b_{2,1}$, then I put $b_{1,3}$, then I put $b_{2,2}$, then I put $b_{2,3}$ and so on. So, the procedure is clear. First list all those elements $B_i j$

where $i+j=2$, then $i+j=3$, then $i+j=4$ so on and so forth. Just list these elements in this order. It is clear that this list exhausts A.

Every single element of the set A will appear somewhere in this list. Of course there will be plenty of repetitions, but that is not a problem, from the exercise any listing, if a set can be listed even with repetitions, it is going to be countable ok. So, this deals with the case where I happens to be denumerable.

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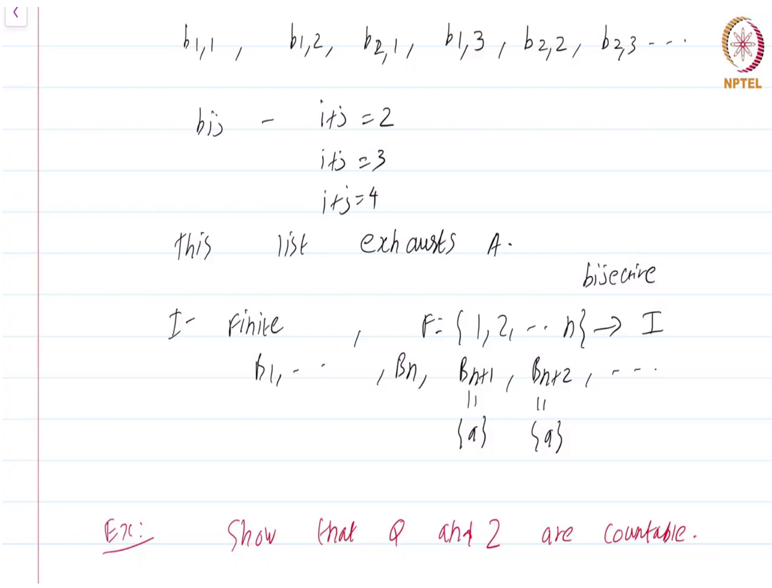

If I is a finite set if I happens to be finite, then we pull the same trick. We have a map F from 1 2 dot dot dot n to I, bijective. So, you will get the sets B 1 to B n. Well then artificially enhance this list to have B n plus 1, B n plus 2 which are all just singleton sets A.

Then repeat the argument we have given above. This proves the result even in the case when I happens to be a finite set and the previous case dealt with it when I was a denumerable set. Hence, we are done. I leave you with one exercise. Show that Q and Z are countable, ok.

This is a straightforward exercise from what we have shown, right now. In the next week's lectures on real numbers, we will see that the set of real numbers has in fact a different cardinality than the cardinality of the natural numbers. That means, you cannot find a bijection from the natural numbers to the real numbers. This is the course on real analysis and you have just watched the module on cardinality.