Real Analysis - I

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Lecture – 35.4
Proof of Weierstrass Approximation Theorem

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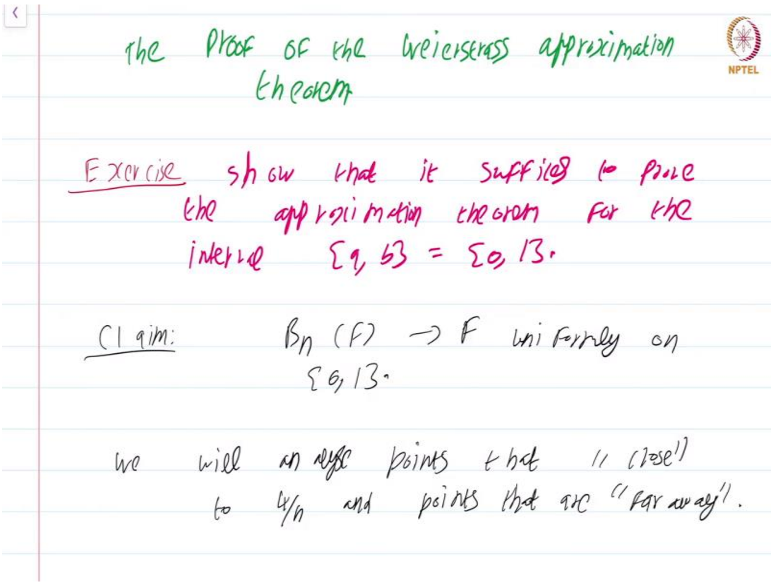

We now give the proof of the Approximation Theorem. So, what we will do is, we will first make a reduction that should have been clear to the reader once during our discussion of the Bernstein polynomials.

So, exercise, show that it suffices to prove the approximation theorem for the interval [a, b] being nothing, but close [0, 1] ok. So, you do not need to establish it for an arbitrary interval, you can consider the special case close interval [0,1] and that is enough ok.

So, once this exercise is done, we will focus on close [0,1] and we will use the Bernstein polynomials ok. So, central claim is that these $B_n(f)'s$ converge to f uniformly on close [0,1]; that is what we have to show ok.

So now, to do this as I had mentioned before we are going to analyze points in two, I mean two separate set of points. What we are going to do is we are going to analyze, we will analyze points that are close to $\frac{k}{n}$ and points that are far away; that is somewhat vague remark, but that is what we are going to do.

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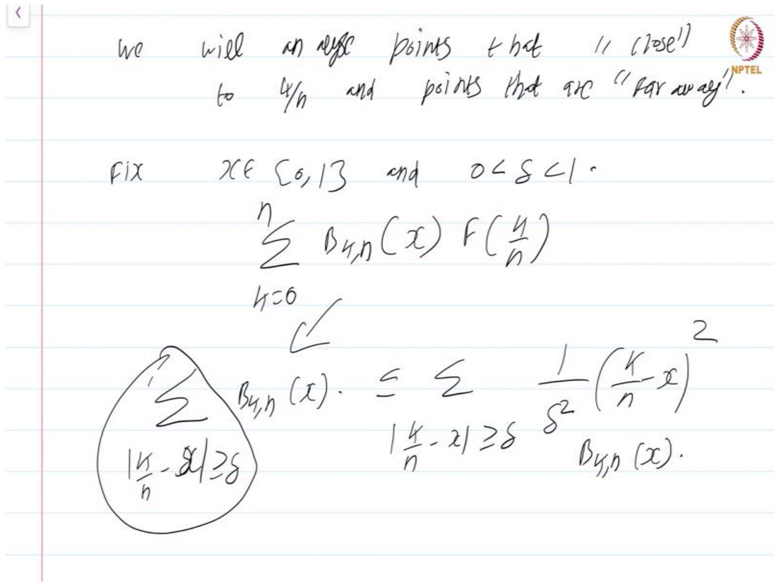

So, let us begin the proof. So, fix $x \in [0,1]$ and $0 < \delta < 1$ ok. Now, we want to analyze the $\sum_{k=0}^{n} B_{k,n}(x) f\left(\frac{k}{n}\right)$. This is what $B_n(f)$'s ok.

So, to analyze this what we will do is, we will just focus on the sum of the Bernstein polynomials. First, what we will do is we will consider $\sum_{|\frac{k}{n} - x| \geq \delta} B_{k,n}(x)$ ok.

So, note we have fixed the x, we want to analyze $\sum_{k=0}^{n} B_{k,n}(x) f\left(\frac{k}{n}\right)$. What we are going to first do is we are going to consider the sum $B_{k,n}(x)$ where $\frac{k}{n}$ is far away from x ok. Note x is fixed ok. Now this is certainly of course, I am using a shortcut here by this is to denote that we are only summing up over those case for which $|\frac{k}{n} - x| \geq \delta$.

Now, this is of course, $\sum_{|\frac{k}{n} - x| \geq \delta} B_{k,n}(x) f\left(\frac{k}{n}\right) \leq \sum_{|\frac{k}{n} - x| \geq \delta} \frac{1}{\delta^2}\left(\frac{k}{n} - x\right)^2 B_{k,n}(x)$ ok. So, what we have essentially done is we have multiplied by $\left(\frac{k}{n} - x\right)^2 \geq \delta^2$ and we are dividing by δ^2.

So, in essence we are multiplying each term $B_{k,n}(x)$ by a quantity that is greater than or equal to 1. So, this is certainly going to be less than or equal to ok.

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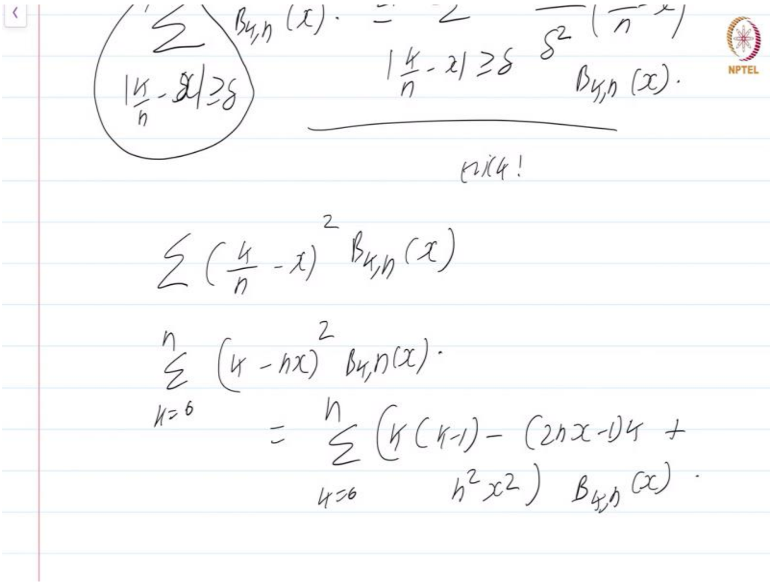

So, now we have to estimate. So, this is a trick. So, this is essentially just the trick. So, now, we want to estimate summation $\sum \left(\frac{k}{n} - x\right)^2 B_{k,n}(x)$ ok; this is what we have to do. So, look at $\sum_{k=0}(\mathrm{k} - nx)^2\, B_{k,n}(x)$, look at this quantity instead ok.

Now, from one of the identities involving the Bernstein polynomials that was left as an exercise to you last time, we can simplify this in a not one by using those properties we can greatly simplify this. What happens is this is, nothing but $\sum_{k=0}^{n}(k(k-1) - (2nx-1)k + n^2x^2)B_{k,n}(x)$, ok.

This whole thing times $B_{k,n}(x)$ ok. So, please check this; please check that you get $(k(k-1) - (2nx-1)k + n^2x^2)B_{k,n}(x)$; once you expand this out.

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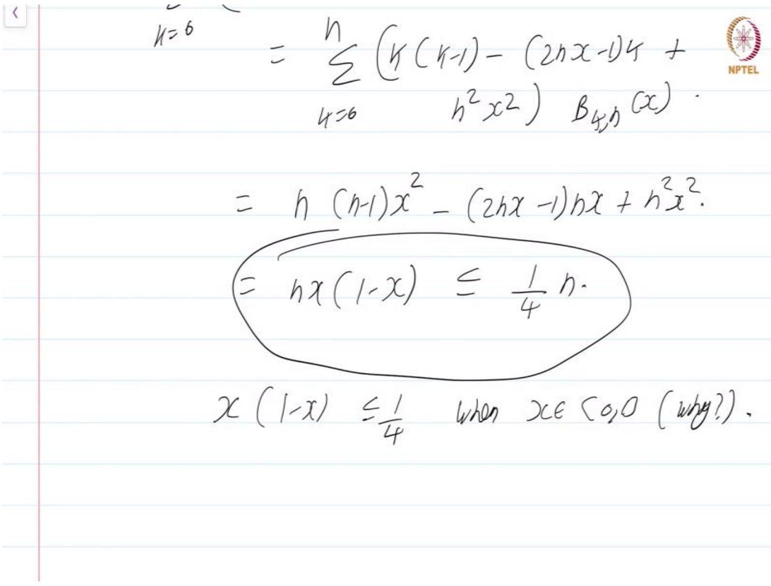

And this by the various identity is that we have established involving the Bernstein polynomials this is nothing, but $n(n-1)x^2 - (2nx-1)nx + n^2x^2$ fine.

So, you can check that this is nothing, but $nx(1-x)$, once you do this a basic arithmetic. And, this is going to be less than or equal to $\frac{1}{4}n$. How did we get this last step? That seems a bit weird, well the last step involves this basic property that $x(1-x) \leq \frac{1}{4}$ when $x \in [0,1]$.

Do you know why this is true? Can you prove this? You can use calculus to prove it, you can also prove it by elementary observations. I urge you to try it in two different ways; one using calculus and one using just elementary basic stuff ok.

So, now, that we have this equation that $nx(1-x) \leq \frac{1}{4}n$. Ultimately what we have is the term we started out with; $\sum \left(\frac{k}{n} - x\right)^2 B_{k,n}(x) \leq \frac{1}{4}n$; what we do is we divide both sides.

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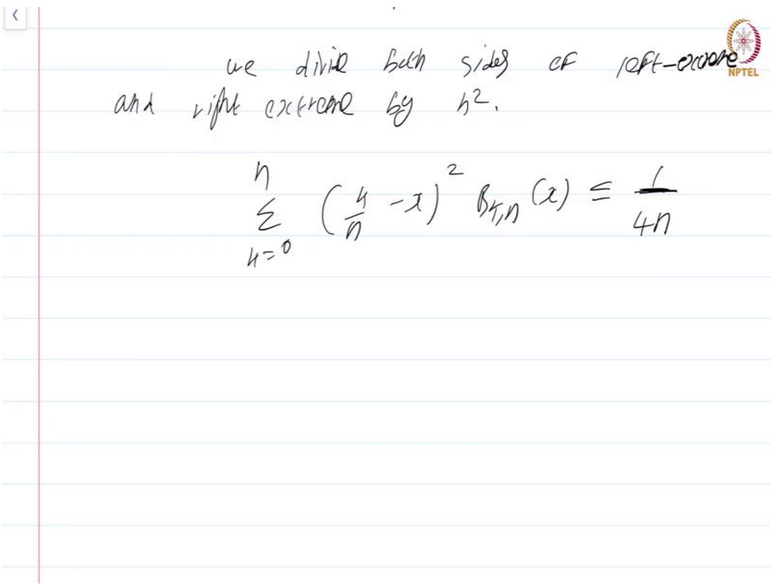

As expected we divide both sides of left extreme and right extreme by n^2 ok. Once we do that we get $\sum_{k=0} \left(\frac{k}{n} - x\right)^2 B_{k,n}(x) \leq \frac{1}{4n}$ ok.

So, what have we managed to achieve? Well, what we have done is we fixed δ and we managed to show that if $\left(\frac{k}{n} - x\right)^2 \geq \delta^2$; in other words $|\left(\frac{k}{n} - x\right)| \geq \delta$. Then, this $\sum_{k=0} \left(\frac{k}{n} - x\right)^2 B_{k,n}(x) \leq \frac{1}{4n}$.

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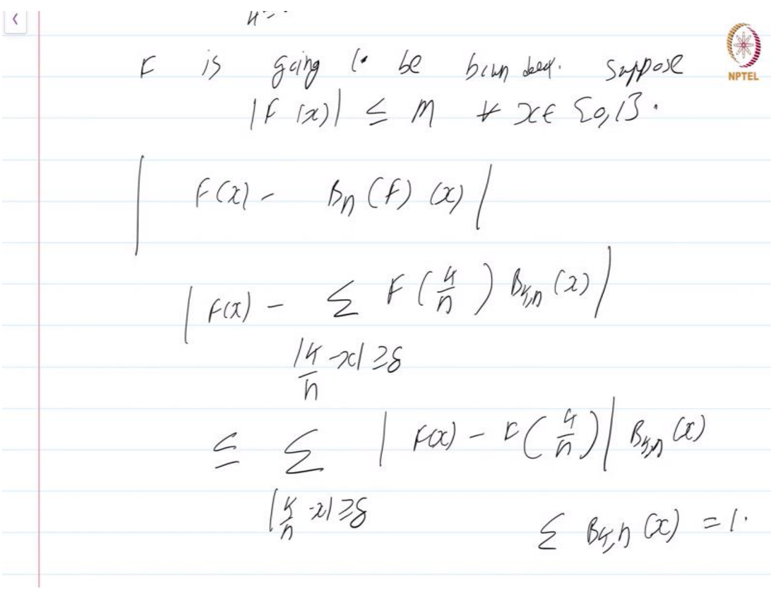

Now, what are we going to do with this? Well, we know that f is going to be bounded. Why? Because, it is a continuous function on a closed interval; so, f is going to be bounded. Suppose,

$|f(x)| \leq M$ for all $x \in [0,1]$ ok. So, choose a sort of an upper bound for this function capital M. Then what the final quantity we are really interested in is $|f(x) - B_n(f)(x)|$.

This is what we are interested in; this is going to be nothing, but $\left|f(x) - \sum_{|\frac{k}{n} - x| \geq \delta} f\left(\frac{k}{n}\right) B_{k,n}(x)\right|$. So, I should not really write equal to, I should write we have first analyzing those terms; we are analyzing those terms for which $|\frac{k}{n} - x| \geq \delta$. So, we are essentially analyzing the paths where $|\frac{k}{n} - x| \geq \delta$ separately.

So, $\left|f(x) - \sum_{|\frac{k}{n} - x| \geq \delta} f\left(\frac{k}{n}\right) B_{k,n}(x)\right| \leq \sum_{|\frac{k}{n} - x| \geq \delta} |f(x) - f\left(\frac{k}{n}\right) B_{k,n}(x)|$ ok. Now, how did we pull this trick? How did we pull this trick? Well, we pulled this trick because summation $\sum B_{k,n}(x) = 1$. This we already know.

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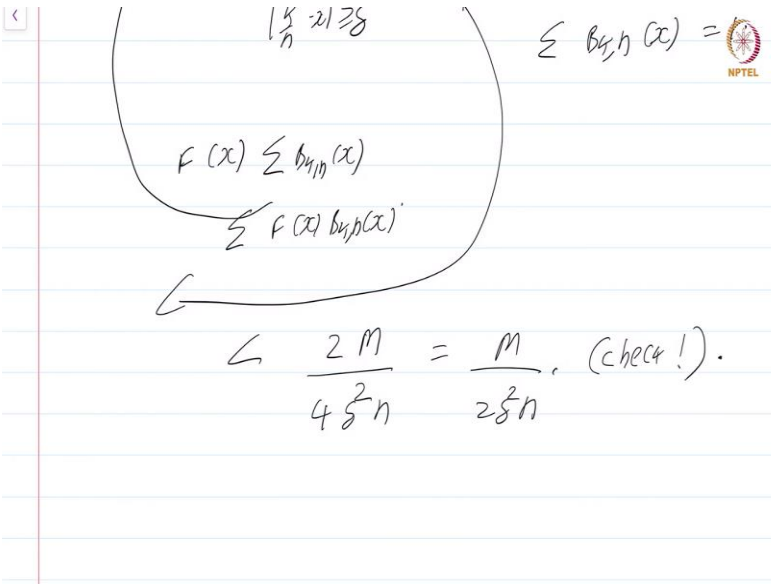

So, what we have essentially done to establish this from the previous step is this $f(x)$, we have written as $f(x) \sum B_{k,n}(x)$. And we have taken this $f(x)$ inside so, we have written this as $\sum f(x) B_{k,n}(x)$. So, this is what we did to the first term, this is what we did to the first term ok. From that this should become apparent how we got $\left|f(x) - f\left(\frac{k}{n}\right)\right| B_{k,n}(x)$ok.

Now, combined with what we have established regarding $\sum_{k=0}\left(\frac{k}{n}-x\right)^2 B_{k,n}(x) \leq \frac{1}{4}n$ and combining that, what we get is this quantity. This quantity is going to be less than $\frac{2M}{4\delta^2 n}$ which is nothing, but $\frac{M}{2\delta^2 n}$, ok. So, kindly check this, kindly check ok.

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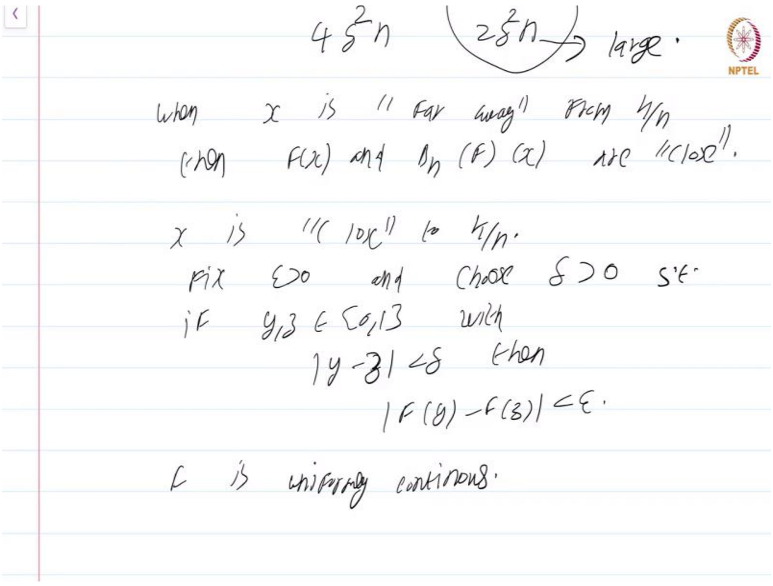

So, what we have now established is that, when x is far away from $\frac{k}{n}$, then $f(x)$ and $B_n(f)(x)$ are close right. All we have to do to make $B_n(f)$ close to $f(x)$, now is to make this small n large. If you make small n large then we will get that B and $f(x)$ are very close to each other.

Now, we have to deal with those points for which x is close to $\frac{k}{n}$} ok. So, what we have essentially done is, we are splitting the $\sum B_k$ and $f(x)$ into two parts; those where which $\frac{k}{n}$ is close to x and those where $\frac{k}{n}$ is far away from x.

So, what we do is, fix $\varepsilon > 0$ and now we are going to choose δ appropriately to get what we need and choose $\delta > 0$ such that, if $y, z \in [0,1]$ with $|y-z| < \delta$. Then $|f(y)-f(z)| < \varepsilon$.

We can do this because, f is uniformly continuous; because f is uniformly continuous, I can find a δ that works universally for all points $y, z \in [0,1]$. Note that this second path analyzing where, x is close to $\frac{k}{n}$ we will not really require any of the properties of the Bernstein polynomials; just the fact that f is uniformly continuous is enough to finish the proof ok.

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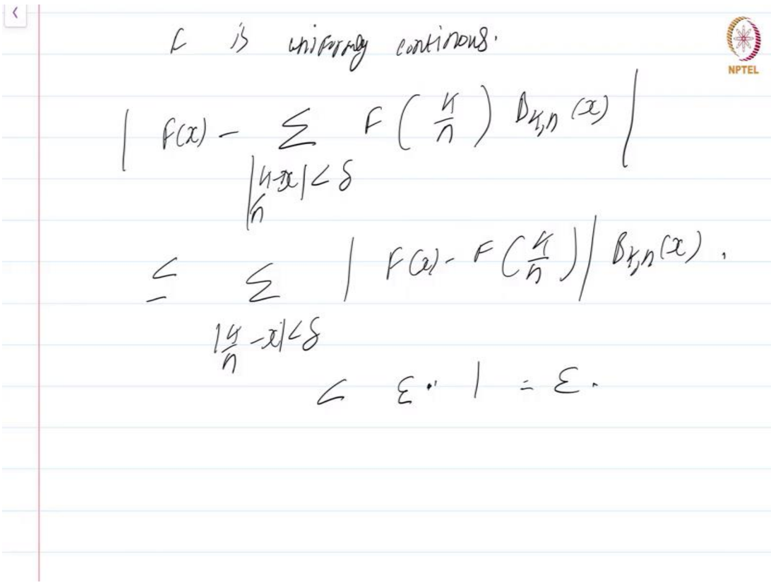

So, now we need to again analyze $\left|f(x) - \sum_{\left|\frac{k}{n}-x\right|<\delta} f\left(\frac{k}{n}\right) B_{k,n}(x)\right| \leq \sum_{\left|\frac{k}{n}-x\right|<\delta} \left|f(x) - f\left(\frac{k}{n}\right)\right| B_{k,n}(x)$

By the exact same argument that I had highlighted; when we had a similar equality, I mean inequality in the previous part ok. But, we are summing up over those $\frac{k}{n}$ where, $\left|\frac{k}{n} - x\right| < \delta$. And, by uniform continuity this is just going to be less than $\varepsilon \cdot 1$ ok, which is ε ok.

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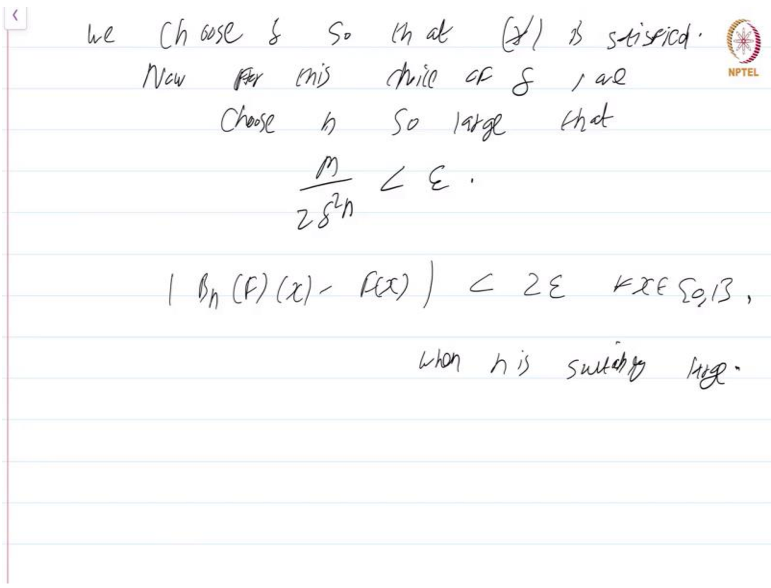

So, now what we do is we choose. So, we have to combine both parts. We choose δ so, that so, that this equation is satisfied. So, that star is satisfied, that is those points where $\frac{k}{n}$ is close to δ can be δ with.

Now, for this choice of δ, we choose n so large that $\frac{M}{2\delta^2 n}$ which was nothing, but the quantity that we got when we analyze the previous term is also less than ε is also less than ε. Net up short is we will get that $|B_n(f)(x) - f(x)| < 2\varepsilon$ for all $x \in [0,1]$ ok.

This will be true when n is suitably large; n is suitably large and this concludes the proof. So, the proof of the Weierstrass approximation theorem that we have given ultimately relies on the uniform continuity of the function f and the basic properties of the Bernstein polynomials.

We analyze those points x which are close to some $\frac{k}{n}$ and for analyzing this just uniform continuity is enough. For those points for which $\frac{k}{n}$ is somewhat far away, we just use the basic properties of the Bernstein polynomials and combining both we get the proof.

This is a course on Real Analysis and you have just watched the module on the Proof of the Weierstrass Approximation Theorem.