Real Analysis - I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 35.3
Properties of Bernstein Polynomials

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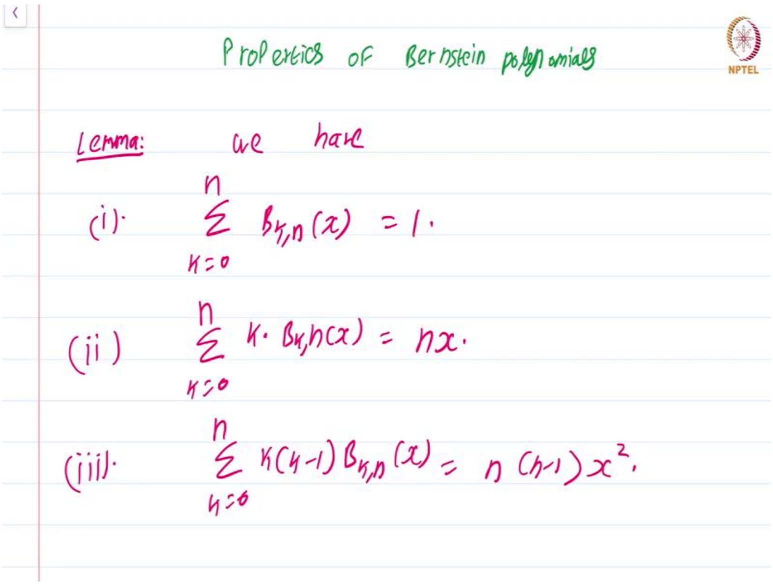

The purpose of this short module is to prove a lemma that captures all the basic Properties of Bernstein Polynomials that will be required in the proof of the Weierstrass approximation theorem. So, let me just state the lemma. Lemma, we have

(i) $\sum_{k=0}^{n} B_{k,n}(x) = 1$. We have already seen a proof of this basic property.

(ii) $\sum_{k=0}^{n} kB_{k,n}(x) = nx$

(iii) $\sum_{k=0}^{n} k(k-1)B_{k,n}(x) = n(n-1)x^2$

So, these are some of the basic algebraic identities involving the Bernstein polynomials that will be needed in the proof of Weierstrass approximation theorem.

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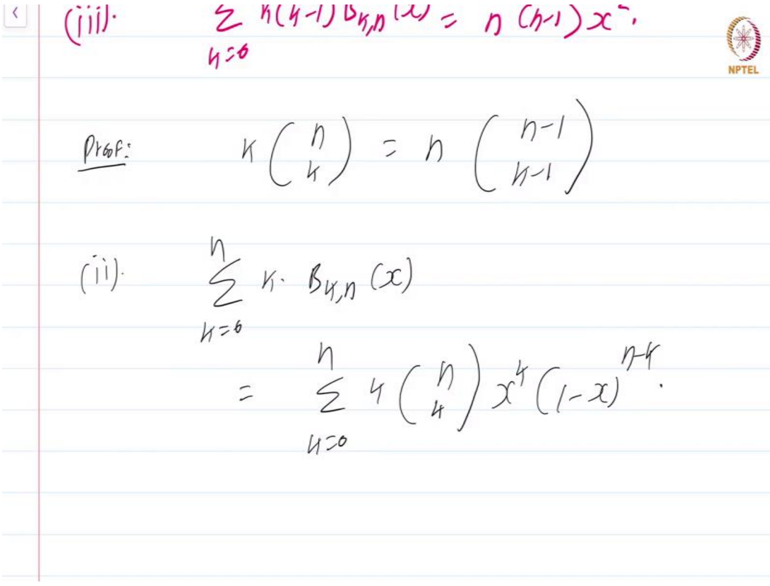

So, let us see the proof. So, for this proof I would need one basic property of these combinations that is I need this fact that $k\binom{n}{k}$ = n$\binom{n-1}{k-1}$. So, I am not going to prove this basic fact, it is just involves expanding out the left-hand side and expanding out the right-hand side and seeing that both sides are actually equal. You might have probably seen this in high school. So, this identity will be needed.

So, let us proceed to proof of the second part; proof of the second part because the first part we have already seen a proof. So, we have to consider $\sum_{k=0}^{n} kB_{k,n}(x)$ We want to analyze this quantity. So, let us just expand it out and write it as $\sum_{k=0}^{n} kx^k(1-x)^{n-k}$ ok. So, I am just summing over all $k's$, I am summing over the expansions.

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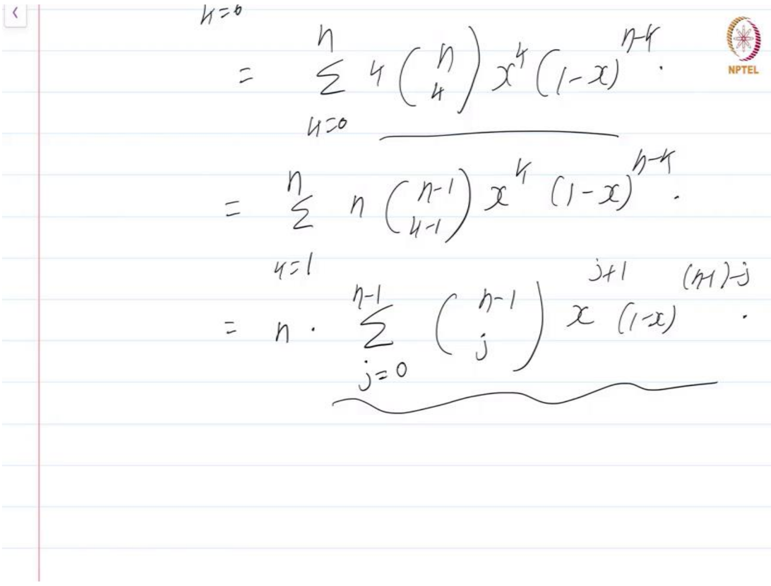

This is nothing, but $\sum_{k=0}^{n} k\left(\binom{n}{k}\right) x^k (1-x)^{n-k} = \sum_{k=1}^{n} n\binom{n-1}{k-1} x^k (1-x)^{n-k}$, note I have used the identity $k\binom{n}{k}$ = n$\binom{n-1}{k-1}$. Note the quantity corresponding to $k = 0$ vanished because obviously, when $k = 0$, this term is 0 ok.

So, now, that we have this, we can take this n outside simply because there it is not dependent on the variable over which we are summing which happens to be k. So, we can write this as $n \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} x^{j+1}(1-x)^{n-1-j}$, all I have done is set $k - 1 = j$, ok. So, all I have done is set $k = j$ +1 fine. So, this is just standard manipulation that you are no doubt familiar with from high school ok.

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$$= n \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} x (1-x)^{n-1-j}.$$

$$= n x \sum_{j=0}^{n-1} \binom{n-1}{j} x^{j} (1-x)$$

$$nx \cdot 1. = nx.$$

identity (iii) is proved in a similar way and is left as an exercise.

Now, look at this $\sum_{j=0}^{n-1} \binom{n-1}{j} x^{j+1}(1-x)^{n-1-j}$ ok. What you do to this is we write this as equal to $nx \sum_{j=0}^{n-1} \binom{n-1}{j} x^{j}(1-x)^{n-1-j}$, ok.

Now, this should look very familiar to you from the first part this is just 1 so, this is $nx \cdot 1$. We are just summing up over all Bernstein polynomials of degree $n-1$ m now ok. So, that sums up to 1. So, you get $nx \cdot 1 = nx$ that is it this concludes the proof of identity two.

Identity three; identity three is proved in a similar way and I leave it to you is proved in a similar way and is left as an exercise; left as an exercise. So, this concludes this short module involving just basic properties of the Bernstein polynomial. This is a course on real analysis, and you have just watched the module on basic properties of the Bernstein polynomials.