Real Analysis - I
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Lecture – 35.2
Bernstein Polynomials

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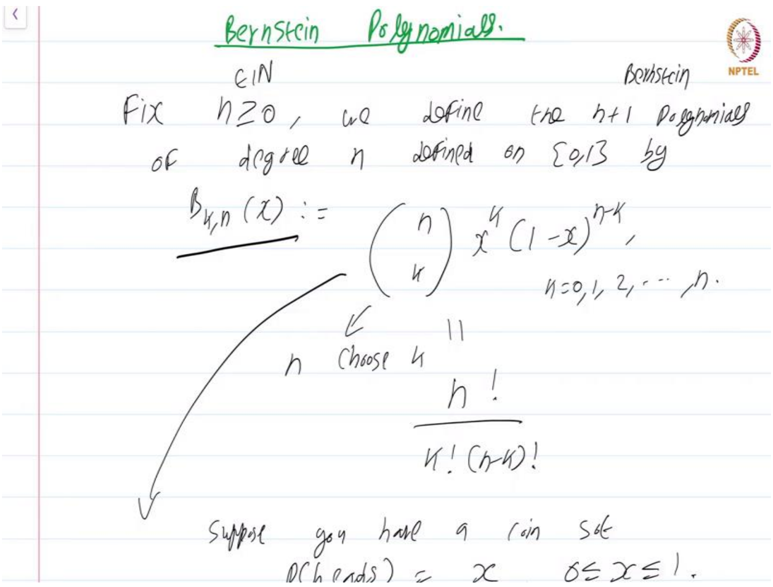

In this module, let us talk about the basic properties of Bernstein polynomials. Of course, we have to begin with what these Bernstein polynomials are. So, fix $n \geq 0$, we define the $n + 1$ polynomials or and the $n + 1$ Bernstein polynomials of degree n defined.

Of course, polynomials are defined on the whole of R, but you will understand why I am saying defined on [0, 1] by $B_{k,n}(x) = \binom{n}{k} x^k (1 - x)^{n-k}$ for $k = 0, 1, 2, \ldots n$.

So, given a fixed number n in the natural numbers, there are these $n + 1$ polynomials each one of them is of degree k and k runs from 0 to n. Now, this is read n choose k, you are familiar with this I am sure from basic probability theory that you have done in your high school, this is just a shortcut for $\frac{n!}{k!(n-k)!}$.

Recall that this $\binom{n}{k}$ as the terminology says is nothing, but the number of ways in which you can choose k objects from a set of n objects ok. So, we have now defined these polynomials, but I have said defined on [0, 1].

Why do I say defined by [0, 1] because if you remember your basic probability, what this actually measures is the following: suppose you have a coin; you have a coin such that probability of heads is x and of course $0 \le x \le 1$ right.

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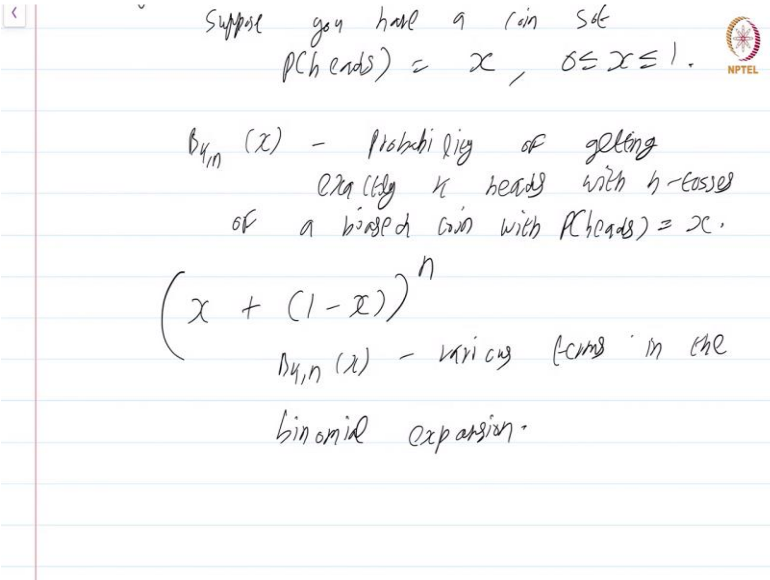

So, in a typical coin, we assume that the probability of heads or tails is equal to half, but you can have biased coins where it falls heads all the time; that means, the probability of heads is 1 or it never falls heads; that means, probability is 0 or anywhere in between. So, this is what is known as a biased coin.

Suppose you have a biased coin such that the probability of heads is actually x, then you should have no difficulty proving from what you have learnt in your high school that $B_{k,n}(x)$ is nothing, but the probability of getting exactly k heads with n tosses of a biased coin with $P(heads) = x$.

So, if this sounds very intuitive and obvious, you can use this to remember what $B_{k,n}$, what the polynomials are, the what the formula for the polynomials are. On the other hand, if you are extremely scared of probability, there is another way to remember what these terms $B_{k,n}(x)$ are just look at $\left(x + (1 - x)\right)^n$. Just look at the whole power n, then $B_{k,n}(x)$ are the various coefficients; various coefficients or rather the various terms not the coefficients various terms in the binomial expansion; in the binomial expansion. This is another way to remember.

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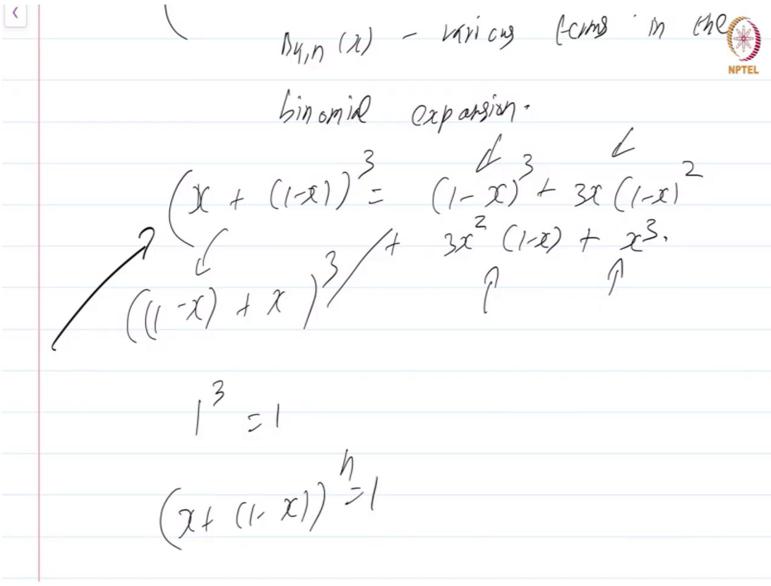

For instance, let us just write down what it is going to be in the case when n = 3. So, you just have $(x + 1 - x)^3 = (1 - x)^3 + 3x(1 - x)^2 + 3x^2(1 - x) + x^3$. So, just look at what $B_{k,n}(x)$ is going to be here, we have $B_{0,3}(x) = \binom{3}{0} x^0 (1 - x)^{n-0}$. So, 3 choose 0 is just 1. So, you will just end up with $(1 - x)^3$ cube which is exactly what you have in the first term.

So, now, that when I think about it, it might be a bit better to write this as $(1 - x + x)^3$, I mean it really makes no difference, but it might be easier to remember if you write it as $(1 - x + x)^3$. So, this way you get these four Bernstein polynomials all of them are obviously of degree 3 ok.

So, why I mean apart from this ease of remembrance, why do you want to write it like this $(x + 1 - x)^3$? Well, $(x + 1 - x)^3 = 1$, right. In fact, in general, $(x + 1 - x)^n = 1$, right.

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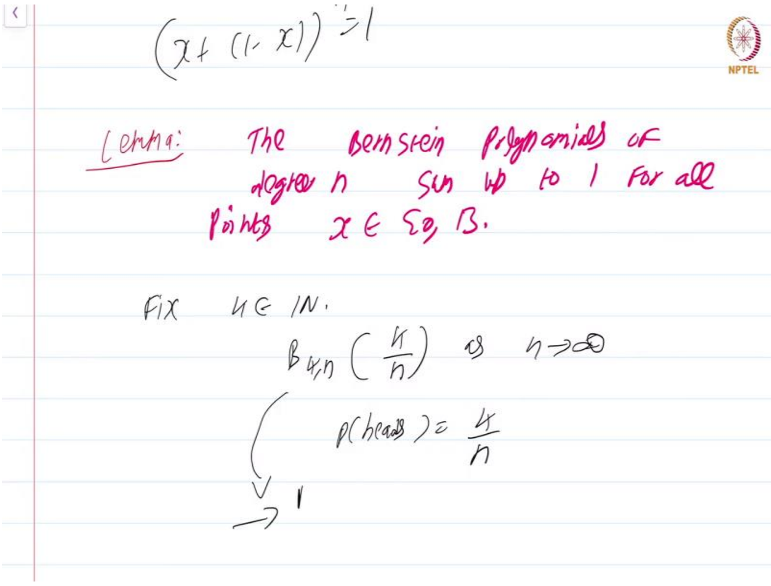

So, we immediately get the following lemma which is so obvious from what we have talked about that I am not even going to bother writing down the proof. Lemma is that the Bernstein polynomials of degree n sum up to 1 for all points$x \in [0,1]$.

Now, again going back to this probabilistic interpretation that $B_{k,n}(x)$ is nothing, but the number of sorry $B_{k,n}(x)$ is the probability that you get k heads when you do n tosses with a bias coin with probability of heads being x, this result is obvious because when you toss a biased coin n times either you are going to get 0 heads or you are going to get 1 head or you are going to get 2 or …. n heads.

One of them is guaranteed to happen. This is just the sum of the probabilities of getting either 0 heads or 1 heads or 2 heads and so on. So, it should obviously, sum up to 1. So, the probabilistic interpretation also gives you another way to give a proof of this lemma that the Bernstein polynomials of degree n sum up to 1 for all points $x \in [0,1]$.

So, we originally started with the goal of approximating an arbitrary continuous function by polynomials. Now, I have introduced some polynomials, these Bernstein polynomials talked about probability and biased coins and n choose k and so on binomial expansions what does this have anything to do with approximating an arbitrary function f? Well, it has something to do with it and that can be seen by this following probabilistic argument.

Now, if you are not familiar with probability, just listen to what I am saying, try to understand if not it is not a big loss. Please revisit this when you get the opportunity to learn probability theory. None of what I say will actually be needed for the proof, but what I am about to say is what is guiding the proof. So, what you do is the following.

Now, fix k. So, let me revert back to black fix k in the natural numbers. What we are now going to do is to use probabilistic intuition to figure out what happens to $B_{k,n}\left(\frac{k}{n}\right)$ as n approaches infinity.

So, what we are essentially doing is we are taking a bias coin not a bias coin we are going to take various bias coins each one of which has probability of heads equal to $\frac{k}{n}$ and analyze what is going to happen if you toss these coins n times and take n to infinity.

Now, if you think about it for a few minutes, you will expect this to converge to 1. Why would you expect this? Well, what is $B_{k,n}\left(\frac{k}{n}\right)$? It is nothing, but the probability that in n tosses you get k heads when you toss a coin whose probability of getting heads is $\frac{k}{n}$, but that merely says that if you take a coin whose probability of landing heads is $\frac{k}{n}$ and if n is very large and you do n tosses, you expect the number of heads to be k.

So, $B_{k,n}\left(\frac{k}{n}\right)$ as the number of tosses increase must converge to 1. So, this is a bit confusing because n is changing therefore, the coins are also changing. So, this is merely an intuition, but it can also be given a rigorous proof and that is known as the weak law of large numbers in probability.

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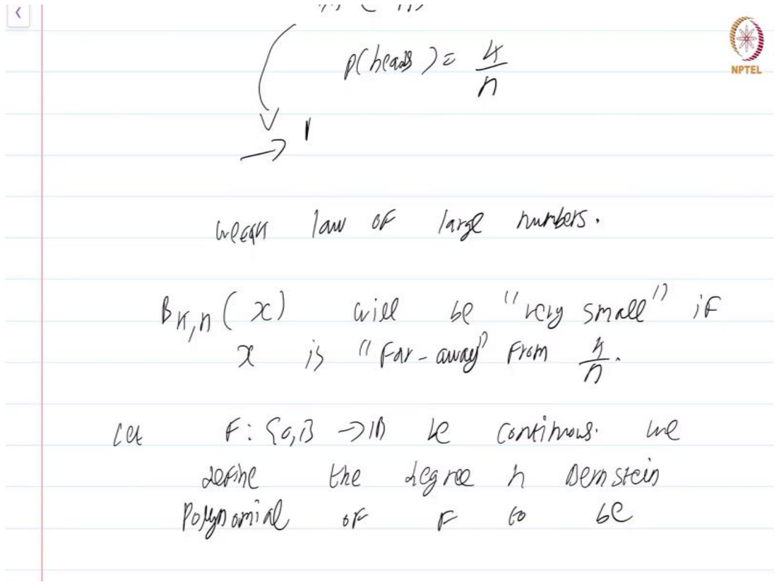

You can justify this rigorously using what is known as the weak law of large numbers and in fact, it is for this reason that Bernstein actually thought that these polynomials could be useful in establishing Weierstrass approximation theorem.

So, if you do not understand why $B_{k,n}(\frac{k}{n})$ must actually converge when it is really not that important because in the proof anyway we will establish that in a self contained manner using just real analysis, but what I am about to say now we will see in very confusing if you do not understand this.

$B_{k,n}(x)$ will be very small if x is far away from $\frac{k}{n}$. So, what this is saying is we saw previously that $B_{k,n}(\frac{k}{n})$ converges to 1 as n tends to infinity. On the other hand, $B_{k,n}(x)$ will be very small if x is far away from $\frac{k}{n}$. Let me put this far away in quotes ok. So, both of these, please think about it in intuitive probabilistic terms, we will anyway establish all this rigorously this is just the basic idea.

So, what we will now do is we will use these Bernstein polynomials and sample values of the function f and use those sampled values to show that you can get a polynomial that converges uniformly to the function f. So, what we do is the following: let $f: [0,1] \longrightarrow R$ be continuous; be continuous. We define the degree n; the degree n Bernstein polynomial; polynomial. Note there is only one such polynomial Bernstein polynomial of f to be as you can guess is just going to be $B_n(f)(x)$.

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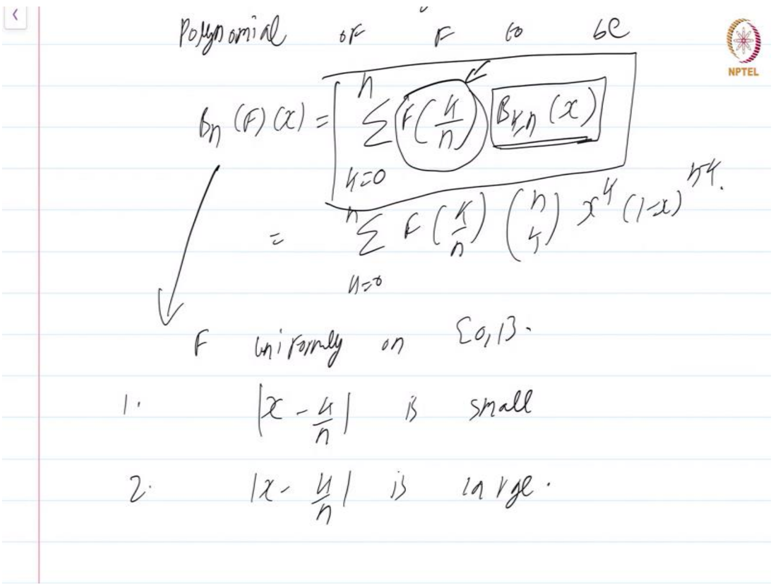

So, essentially you are going to sum up over all the Bernstein polynomials of degree n, you sample the values of f at $\frac{k}{n}$ and multiply it by $B_{k,n}(x)$ which is nothing, but $\sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x)$ and expanding out $B_{k,n}(x)$ you get $\sum_{k=0}^{n} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$. These polynomials $B_n(f)$ are the Bernstein polynomials of degree n.

So, what does our probabilistic intuition tell us? Well, if you look at this sum; if you look at the sum for any value x what will happen is as you increase; as you increase n, then from the intuition that we have seen before this $f\left(\frac{k}{n}\right)$, this value is getting weighed by $B_{k,n}(x)$.

When x is very very close to $\frac{k}{n}$, this quantity $B_{k,n}(x)$ will be very very close to 1 and the value $f\left(\frac{k}{n}\right)$, will be completely got whereas, for those $\frac{k'}{n}s$which are far away from x, these terms $B_{k,n}(x)$ will be 0 and they will contribute nothing to the sum ok. So, essentially, we will get the sampled value of $f\left(\frac{k}{n}\right)$,. So, this is all sounding a bit vague, but do not worry we will make it precise.

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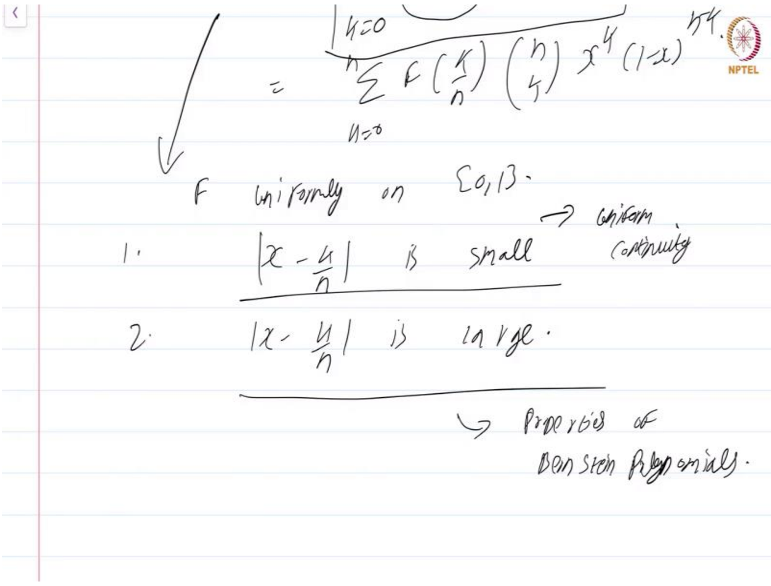

So, what this discussion suggests is that to complete the proof of course, if it is not clear, we are going to prove that $B_n(f)(x)$ converges to f uniformly on $[0,1]$; that is the aim. What we will do is we will split our analysis into two parts; split our analysis into two parts, we will study where $|x - \frac{k}{n}|$ is small and number 2 where $|x - \frac{k}{n}|$ is large. We will split our proof into two parts and handle these two parts separately ok.

In fact, for the 1st part just uniform continuity will give us what we need. Recall that a function defined on a closed interval is automatically uniformly continuous. So, f is uniformly continuous. For the 2nd part, we need some properties; we need some properties of Bernstein polynomials.

Some special properties of Bernstein polynomials are required to handle this case when $|x - \frac{k}{n}|$ is large. As you can expect, we have just done everything very vaguely and intuitively so, it might look like the proof is over, but actually we have to do some analysis to make my vague remarks precise.

So, this is the motivation behind the proof, if you do not understand substantial amount of this, it is not at all an issue because the proof is not going to actually require this intuition, the proof is self contained, but the ideas behind the proof are motivated by the institution what I have spoken about in this module.

In the next module, let us prove some basic identities of the Bernstein polynomials that will be needed in the final proof.