Real Analysis - I
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Lecture – 35.1
Weierstrass Approximation Theorem

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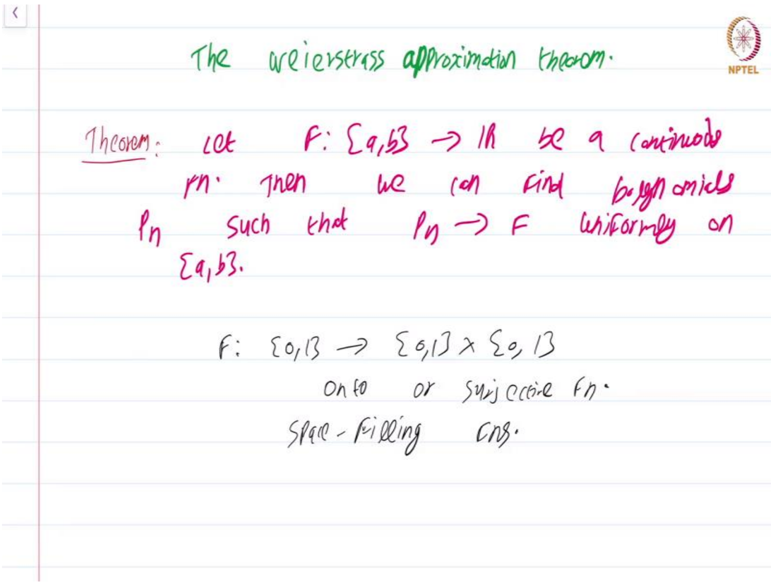

We are at the grand finale of this course and we are going to end with the bang. We are going to study the famous and extremely powerful Weierstrass Approximation Theorem that states that, any continuous function can be approximated uniformly by polynomials. This theorem is extensively used in mathematics and also it is a starting point of the theory of numerical analysis, especially the portion of numerical analysis that deals with polynomial interpolation.

Without further ado, let me just state the theorem, then talk about it a little bit more. Theorem :let $f: [a, b] \longrightarrow R$ be a continuous function then we can find polynomials p_n, such that p_n converge to f uniformly on the closed interval $[a, b]$.

In essence what this theorem is trying to tell us is that, no matter how complicated the function f is; we can always find a polynomial that is as close to f as we desire. So, in all practical applications, we are never interested in getting the exact values of functions. Let us say you want to launch a rocket from earth to the moon, there are certain angles that you will have to take care of; obviously you have control only to some finite degree how much you can angle the rocket.

All you require is to determine the angle to that specified level of error or tolerance, so that the rocket takes the correct course and lands on the moon and does not end up lost in space. So, in all practical applications, we need only a specified degree of approximation. And the Weierstrass approximation theorem guarantees that, no matter how small that tolerance level is; we can always find a polynomial approximation of the function f that is better that satisfies that tolerance level.

So, when you are computing real analytic functions like sine or cosine or exponential, then this is not really an issue; because we know that the Taylor series does the job for us, we can use the finite truncation of the Taylor series to get the required polynomials. But we are now talking about arbitrary continuous functions, and arbitrary continuous functions can be extremely complicated and bizarre.

In the future, I am going to be talking about functions of several variables in the real analysis II course; there we will see that we can find a continuous function $f: [0,1] \longrightarrow [0,1] \times [0,1]$. So, I have not talked about what is the meaning of a continuous function in this setting; yet we will do so in real analysis II in the future, such that this function is onto or surjective.

A surjective function; surjective function from $[0,1] \longrightarrow [0,1] \times [0,1]$ is possible; this is called a space filling curve. So, continuous functions can be extremely bizarre, can be extremely bizarre to expect that, you can approximate any bizarre continuous function by polynomials it itself astounding. So, you can understand why this theorem is so powerful and so useful; we can deal with extremely bizarre and complicated objects by using extremely simple polynomials.

Remember that a polynomial can be quickly computed by a computer; because the base operations in most instruction sets of computers involve addition and multiplication, and a polynomial is just repeatedly doing additions and multiplications. So, computers can evaluate polynomials really quickly, ok. So, the importance of this theorem should now be apparent.

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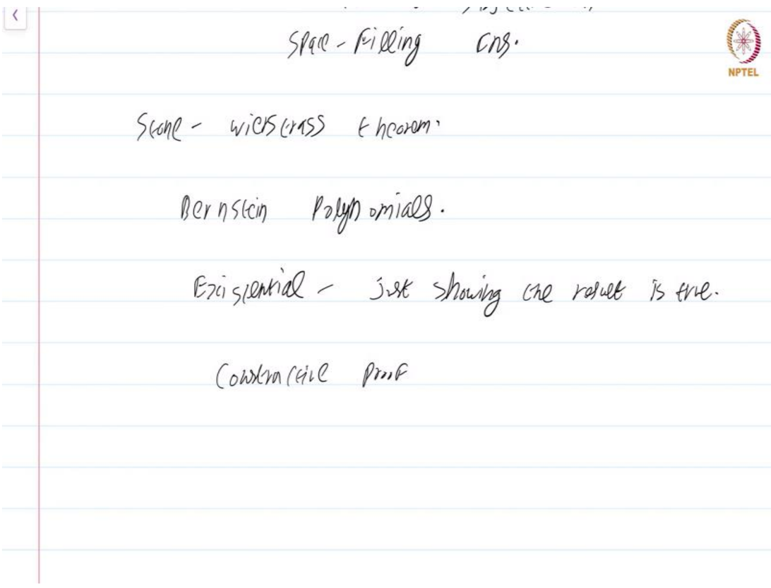

Now, how are we going to prove this theorem? Well, there are several proofs of this theorem; in real analysis II, I will prove what is called the stone Weierstrass theorem, which gives a very general version of the Weierstrass approximation theorem, but that is for a future course. So, I will not talk about it much. In this course, we will give a very simple proof; but it is a bit tricky using what are known as Bernstein polynomials, using Bernstein polynomials.

So, this proof was given by Bernstein and this proof is very elegant and very simple and you it uses simple ideas from probability theory. Since I am not going to assume that you are familiar with probability theory; what I will do is, I will give a completely self-contained proof.

But pass a few comments here and there, such that if you do not know probability theory, you do not lose anything. But at the same time if you know probability theory; you will be able to understand exactly how Bernstein was able to come up with this proof.

One more remark why I have chosen this proof up, there are four or five; why I have chosen this particular proof for this theorem is that, I emphasize that this Weierstrass approximation theorem is used extensively to compute continuous functions to a great degree of accuracy. Now, this is all meaningless if the proof I give you is an existential proof.

What do I mean by an existential proof? Existential proof is just showing that the result is true, just showing the result is true. What I mean by that is; I do a bit of analysis and say that yes of course there are polynomials, which do the job. But this gives you no idea how to compute those polynomials, right.

for all practical applications, not only do you want to know that there exists a collection of polynomials that can be used to approximate the given function f to as good a degree as you want; but you want to actually lay your hands on these polynomials.

The Bernstein polynomials allow you to do this; given a function f, if you know the values of the function at a pre specified set of points, you can actually compute and write down this polynomial. So, you need data about the function only at some points and you actually get a very good approximation of this function.

So, the motivation for approaching this proof via Bernstein polynomials is to have what is known as a constructive proof. A proof that not only gives you the fact that there exist polynomials that do the job, but actually how to get those polynomials if you have some data about the function. So, on to the next module, where I introduce the basic properties of Bernstein polynomials.