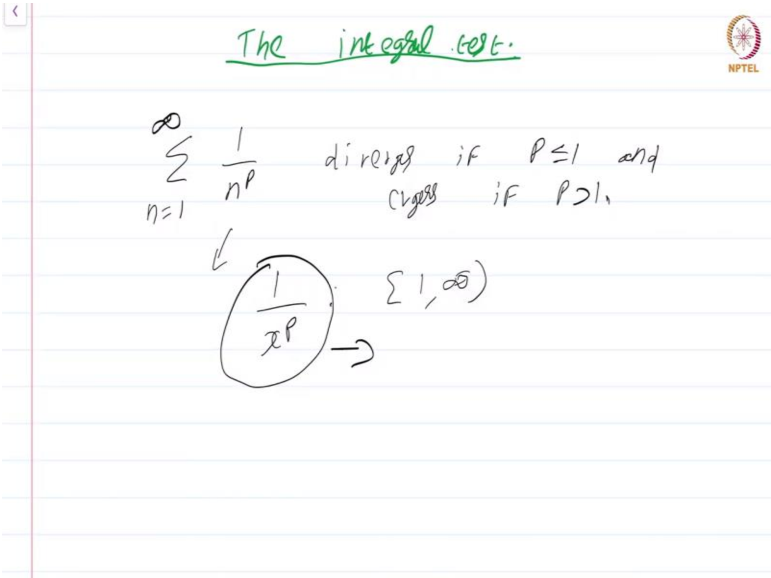
Real Analysis - I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture - 34.2
The Integral Test

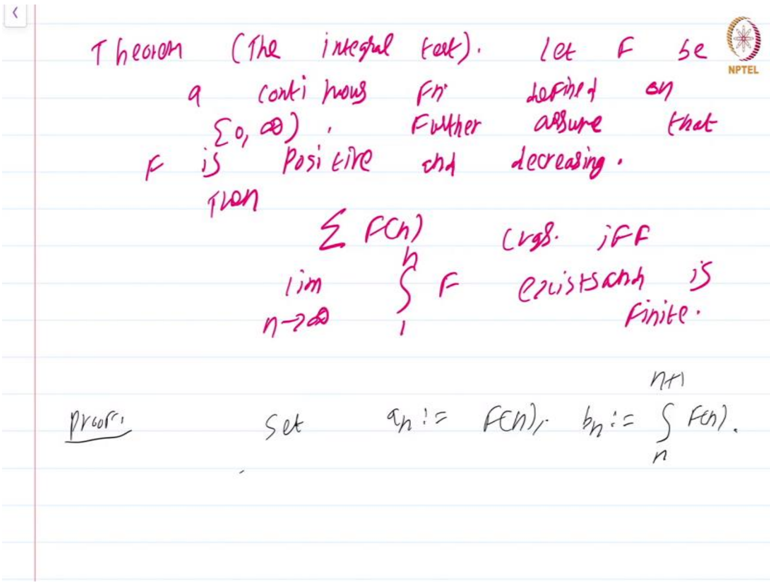
(Refer Slide Time: 00:14)



Recall the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, if $p \leq 1$ and converges if $p > 1$. Now, observe this function $\frac{1}{n^p}$, I can consider it as $\frac{1}{x^p}$ and the function is now defined on $[1, \infty)$ ok. I can extend $\frac{1}{n^p}$, to the whole real numbers in $[1, \infty)$.

Now, it is natural to study whether by using this function $\frac{1}{x^p}$, whether we can conclude something about the series$\sum_{n=1}^{\infty} \frac{1}{n^p}$ and that is precisely what the integral test tries to tell us. It sort of says that the sum of the series is related to the area of this extended function between n and $n + 1$ ok.

(Refer Slide Time: 01:36)

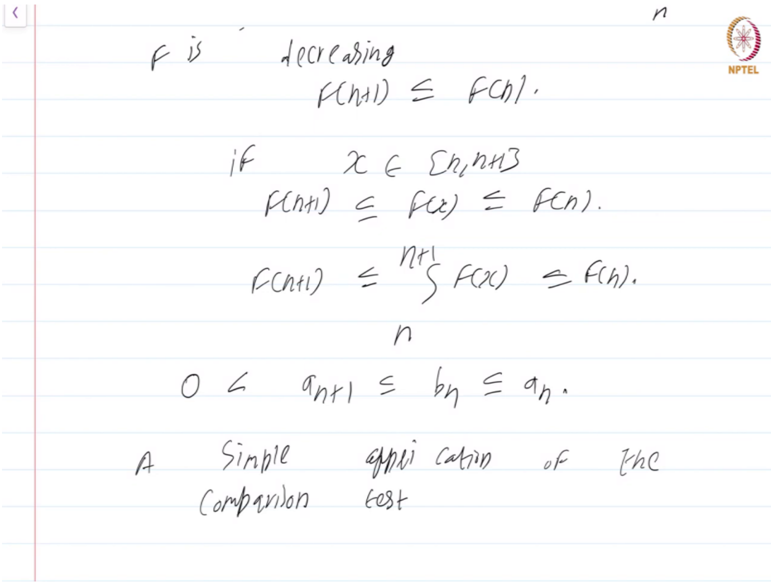


So, these vague remarks will now be made precise in this theorem. This is the integral test and this is a very powerful general test for convergence. Let f be a continuous function, continuous function defined on $(0, \infty)$ ok.

further assume, that f is positive and decreasing ok. Then, $\sum f(n)$ converges if and only if $\lim_{n\to\infty} \int_1^n f$ exists or in other words, the series $\sum f(n)$ converges if and only if the improper integral exists and I must add to be precise and is finite.

Remember, when we defined the improper integral, we did allow the possibility that the integral could be $\pm\infty$. We do not want that. That is obvious. Let us see a proof and my vague remarks about the area between n and $n+1$ and the value of the series being related will become clear in the proof ok. So, proof; set $a_n = \ f(n)$ and set $b_n = \int_n^{n+1} f(n)$ ok.

(Refer Slide Time: 03:41)

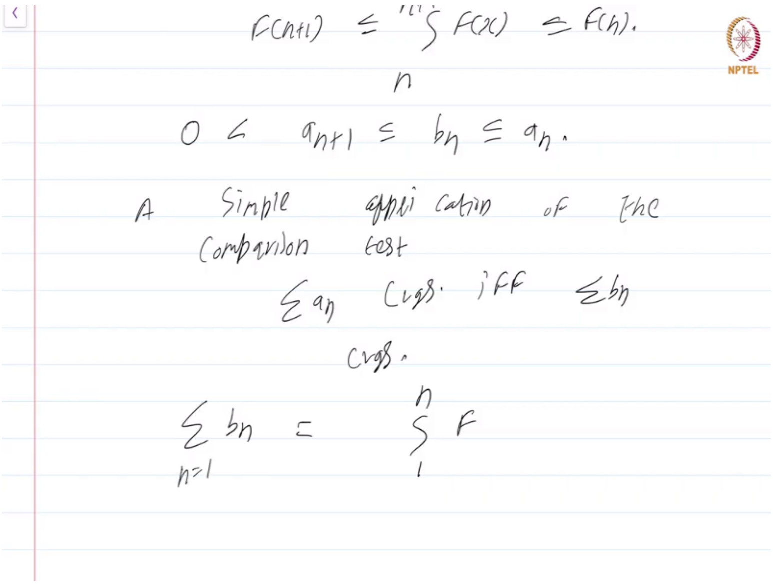


Now, we are assuming that f is decreasing which just means that $f(n+1) \leq f(n)$, that is what decreasing will imply ok. Also, it actually says that if $x \in [n, n+1]$, then $f(n+1) \leq f(x) \leq f(n)$, right, that is what decreasing means ok.

In other words, if I integrate from n to n+1, this is set of inequalities, what I would get is $f(n+1) \leq \int_n^{n+1} f(x) \leq f(n)$, Obviously, integrating the entire equation from n to n+1 that just means that the area of the function or area of the region under the graph of the function from n to n+1 is dominated by $f(1)$ and it dominates $f(n)$; sorry is dominated by $f(n)$ and dominates $f(n+1)$, ok.

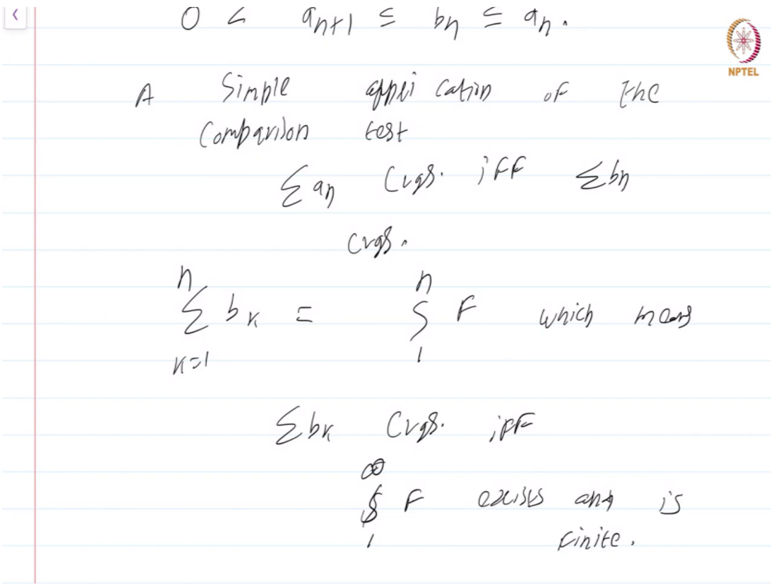
Now, what I can do with this is translate it in terms of a_n's and b_n's. So, what we get is $0 < a_{n+1} \leq b_n \leq a_n$, ok. So, b_n is sandwiched in between a_{n+1} and a_n ok. Now, a simple application of the comparison test, you have to apply it twice, once for each inequality $a_{n+1} \leq b_n$ and $b_n \leq a_n$.

(Refer Slide Time: 05:44)



We get $\sum a_n$ converges if and only if, $\sum b_n$ converges. This just follows immediately if you apply the comparison test twice to the inequality that we have ok. But what are $\sum b_n$? $\sum b_n$ is nothing but $\int_1^n f$ right. Summation not summation b_n; the summation n = 1 or rather let me be a 100 precise.

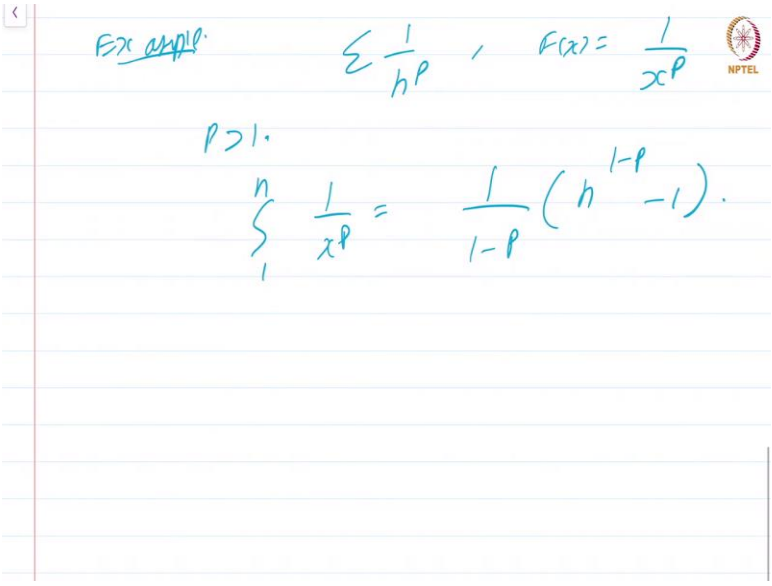
(Refer Slide Time: 06:21)



$\sum_{k=1}^{n} b_k = \int_1^n f$, which means $\sum b_k$ converges if and only if, $\int_1^n f$ exists and is finite. In other words, the convergence of the series $\sum a_n$ is determined by whether the improper integral $\int_1^\infty f$

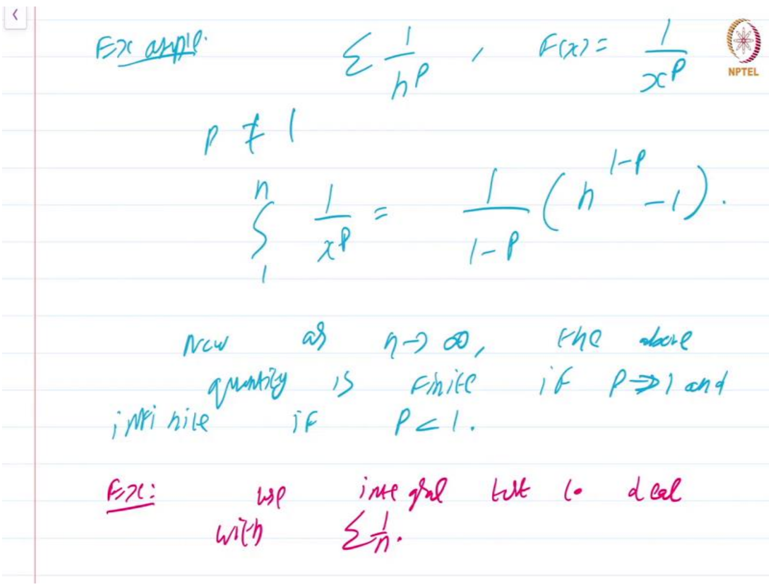
converges or diverges. So, the proof is rather simple and uses the simple idea that the graph, the area under the graph is related to the terms of the series. Now, using the integral test, we can easily show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ does converge if p>1.

(Refer Slide Time: 07:24)



Let us see how this is done. So, example; consider $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Now, choose $f(x) = \frac{1}{x^p}$. Let us first take $p > 1$ and see what happens. Then, $\int_1^n \frac{1}{x^p} = \frac{1}{1-p}(n^{1-p} - 1)$ ok. In fact, for this, we do not need to assume p greater than 1.

(Refer Slide Time: 08:01)



We just need to assume that $p \neq 1$; when $p = 1$, what I have written is not correct, $\int \frac{1}{x} = \log x$ and not the formula that I have written. Now, as n approaches infinity the above quantity, the above quantity is finite, if $p > 1$; and infinite if $p < 1$ ok. That immediately gives that when $p > 1$, the series converges by the integral test and when $p < 1$, we immediately get that the series diverges.

Of course, for the $p = 1$ case, we have to do a special argument which we have already done in an earlier chapter to show that the series $\sum \frac{1}{n}$. So, we were able to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ except the $p = 1$ case, whether it converges or diverges, we were able to show it pretty easily using the integral test ok. So, can you show using the integral test that $\sum \frac{1}{n}$ diverges? Yes, you can and that is going to be left as an exercise.

Use integral test integral test to deal with $\sum \frac{1}{n}$ also. So, this concludes this module. This is a course on real analysis, and you have just watched the module on the integral test.