Real Analysis - I
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Lecture – 34.1
Improper Integrals

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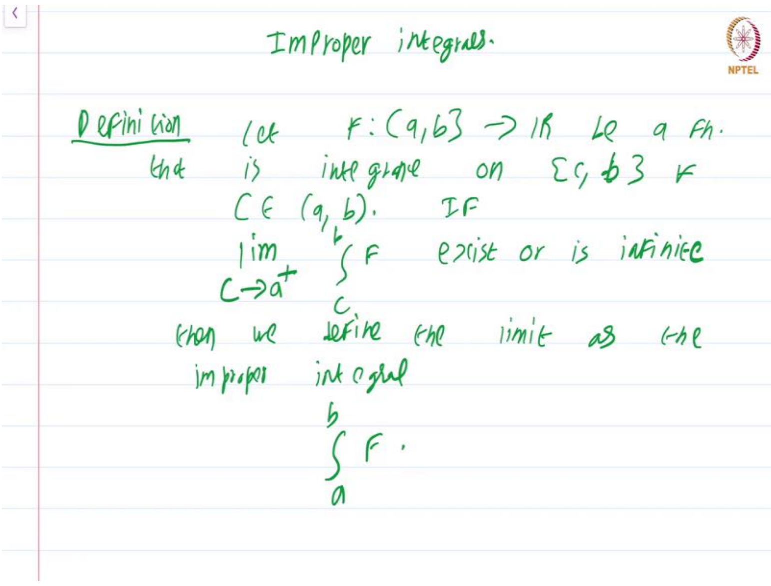

Functions that are bounded and defined on a closed interval are the functions that we have considered in the definition of the Riemann integral. However, in many practical applications, it might be good to have a version of the integral that works for more general functions.

Such integrals can be defined using sophisticated tools for instance the famous Lebesgue integral works for a much larger class of functions than those that are bounded and defined on a closed interval.

However, the Lebesgue integral is quite difficult to define and is technically a part of a graduate course. What we are going to do now is to take a much more restrictive approach and define integrals for certain functions that might be unbounded or defined on an unbounded interval.

This definition is going to be just a slight tweak of the definition of the Riemann integral and is therefore, much easier to handle. As an application, we will use this more general integral to devise a test of convergence of series called the integral test. So, let us move on to the definition.

Definition: let $f: (a, b] \longrightarrow R$ be a function that is integrable on [c, b] for all $c \in (a, b)$. So, you are given a function defined on a one sided interval, the point a is not really there in the interval. However, when you take any point c in the open interval (a, b) and restrict this function f to the closed interval $[c, b]$, you end up with an integrable function ok.

If $\lim\limits_{c \to a}$,I should say $\lim\limits_{c \to a^+}$; because technically I want c to come only from the closed interval [a, b]. If $\lim\limits_{c \to a^+} \int_c^b f$ exists or is infinite, then we define; the limit as the improper integral; $\int_a^b f$ ok.

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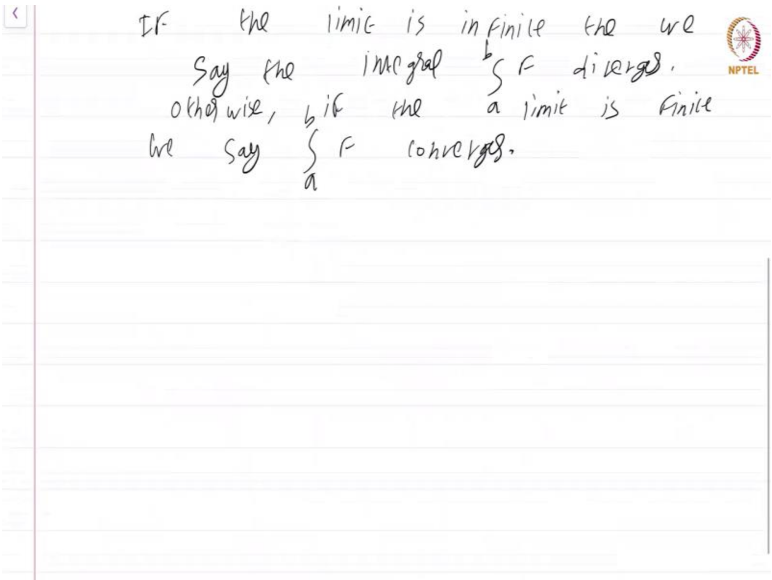

Let me just add a remark. If the limit is infinite, it could be + or - infinity. If the limit is infinite, then we say; the integral $\int_a^b f$ diverges. Otherwise, if the limit is finite we say integral $\int_a^b f$ converges ok.

So, note carefully $\int_a^b f$ could not exist at all in the first place, then if it exists it could either converge or diverge. If it converges, then the value is going to be a finite real number. If it diverges, it is going to be either + or - infinity ok.

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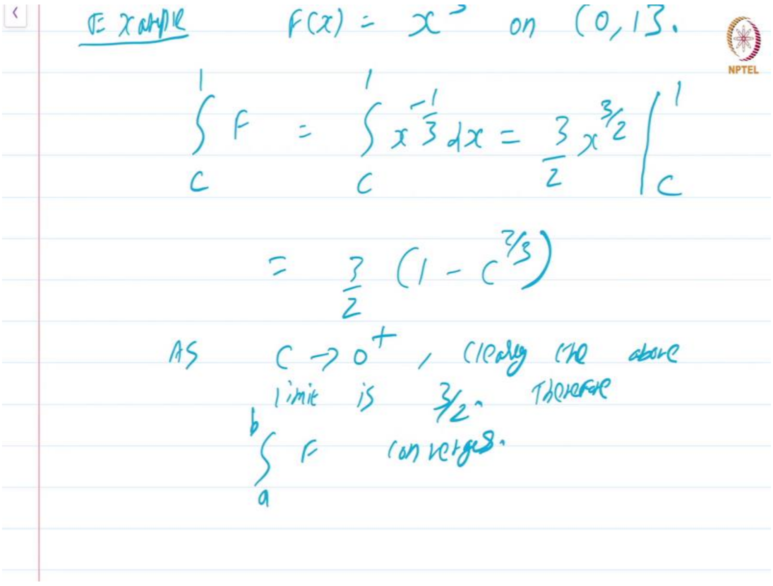

So, where does such integrals come up in practice? Well, the best way to see that is to actually see an example of such an integral. Example: let us consider the function $f(x) = x^{\frac{-1}{3}}$ and we define this on the interval open $(0,1]$ ok. Note that this function is not even defined at 0.

Now, let us see whether we can make; I mean we can compute the improper integral of this function $x^{\frac{-1}{3}}$. Well, let us integrate $\int_c^1 f = \int_c^1 x^{\frac{-1}{3}}\, dx$ and from elementary high school integration this is just $\frac{3}{2}x^{3/2}\Big|_c^1$ and I have to substitute the limits which is c and 1; which is just going to be $\frac{3}{2}\left(1 - c^{\frac{2}{3}}\right)$ ok.

Now, as c approaches 0 + clearly; the above limit is $\frac{3}{2}$; above limit is $\frac{3}{2}$ ok. Therefore, integral $\int_a^b f$ converges. So, the improper integral of $x^{\frac{-1}{3}}$ on the interval (0, 1] exists and in fact, this integral converges, and the value of the integral is going to be $\frac{3}{2}$.

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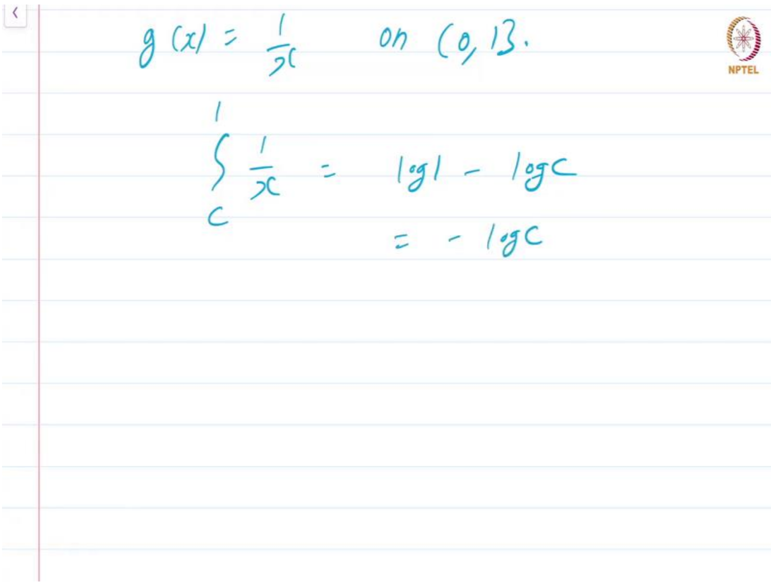

Now, let us see an example where the improper integral diverges. for this take $g(x) = \frac{1}{x}$, again defined on $(0,1]$. Just judging from the graph, you would think that this integral; improper integral does not exists, but let us check that rigorously.

Again, we have to compute integral $\int_c^1 \frac{1}{x}$. Again, from elementary calculus that you have learnt in school this is just $\log 1 - \log c = -\log c$, because log 1= 0.

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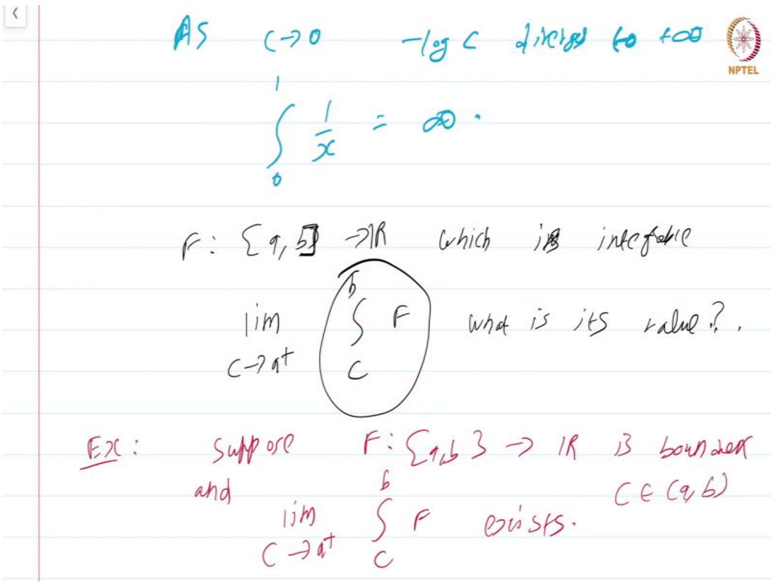

Now, as c approaches 0, this - log c sort of diverges to $+\infty$ ok. So, in other words, integral $\int_c^1 \frac{1}{x} = +\infty$ ok. Now, if you have been following what I am saying carefully, the following question would have arisen to you. Suppose I start with a function; start with a function $f: [a, b] \longrightarrow R$ and let me write it as closed so that there is no confusion it is closed $[a, b]$ to R which is integrable.

Suppose I take such a function, then there is absolutely nothing stopping me from considering $\lim_{c \to a^+} \int_c^b f$ ok. Think about why integral $\int_c^b f$ is also defined it is rather easy why it is also defined. So, look at $\lim_{c \to a^+} \int_c^b f$ what is its value ok? So, to understand this, I am going to give you an exercise.

Exercise: Suppose; $f: [a, b] \longrightarrow R$ is bounded and $\lim_{c \to a^+} \int_c^b f$ exists. Note implicitly when I write $\lim_{c \to a^+} \int_c^b f$, I am assuming that integral $\int_c^b f$. In other words, for all point c in open interval (a, b), I am assuming that the function f when restricted to closed interval [c, b] is integrable not only is it integral I am assuming that the limit exists.

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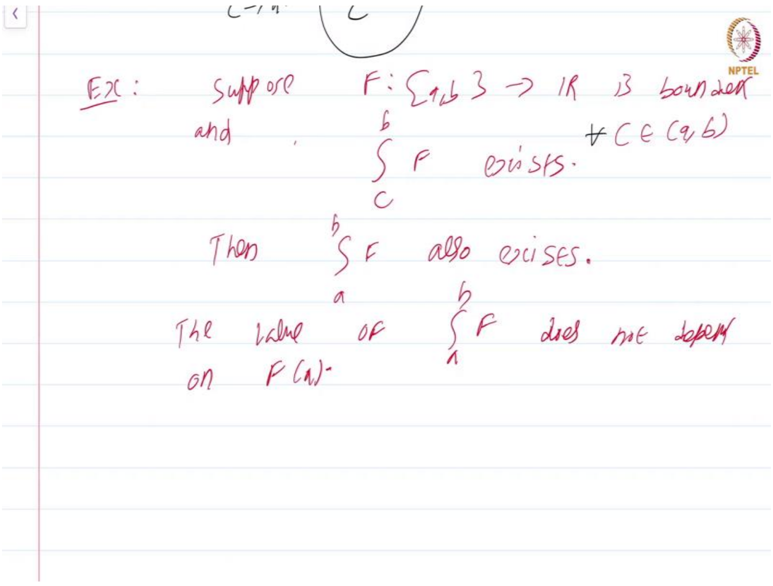

Then, integral $\int_a^b f$also exists; also exists ok. Now, that I think about it I do not need to put this extreme restriction and I can just assume integral $\int_c^b f$ exists for all c; for all $c \in (a, \ b)$ that is enough. So, if you have a closed interval $[a, b]$ and a bounded function f on that closed interval

$[a, b]$ and suppose you have that for all point c in between a and b integral $\int_c^b f$ exists, then integral $\int_a^b f$ also exists.

So, in short, what this is saying is the improper integral you need to tackle the special case of improper integral only under the condition that the function f is not actually bounded on the interval (a, b). So, to clarify that let me also remark the value of this integral $\int_a^b f$ does not depend on; does not depend on $f(a)$.

You can set $f(a)$ to whatever you want; the value of this integral is continued to be unchanged ok. So, please solve this exercise. It is important and let you understand why exactly we are defining improper integrals the way we are ok.

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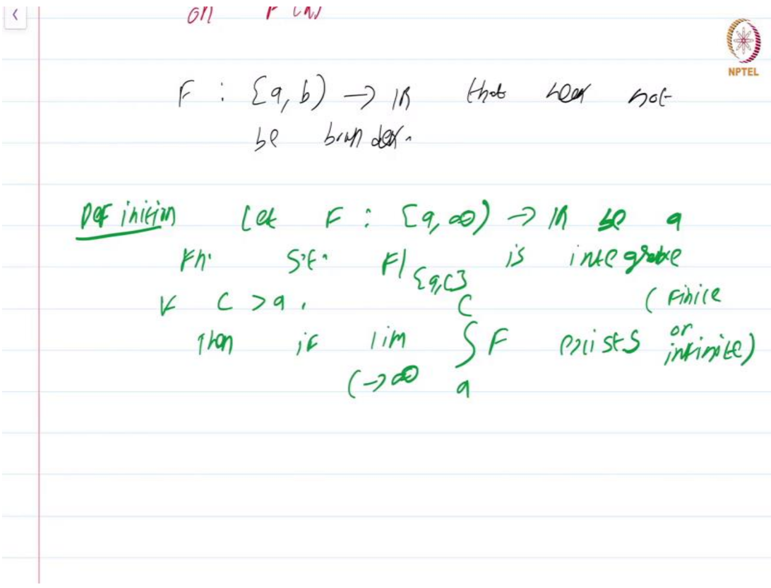

Now, there is no difficulty in generalizing this notion of improper integral to intervals of this form $f: [a, b] \longrightarrow R$ that need not be bounded, it is exactly analogous that need not be bounded. You can define for just like the left end point be not being there in the definition of f and the function being unbounded, you can consider $f: [a, b) \longrightarrow R$ in the exact same way.

One other case remains that is functions that are defined on infinite intervals. Definition; let $f: [a, \infty) \longrightarrow R$ be a function such that $f: [a, c] \longrightarrow R$ or rather I will not use this notation $f|_{[a,c]}$ is integrable for all $c > a$ ok.

So, what I am doing is I am considering a function that is defined on one sided infinite interval such that whenever you take a finite quantity c and restrict the function to the closed interval $[a, c]$ the function f is integrable. Then if; $\lim_{c \to \infty} \int_a^c f$, you should guess where this is integral $\int_a^c f$ exists ok, by exist I could mean it could be finite or infinite.

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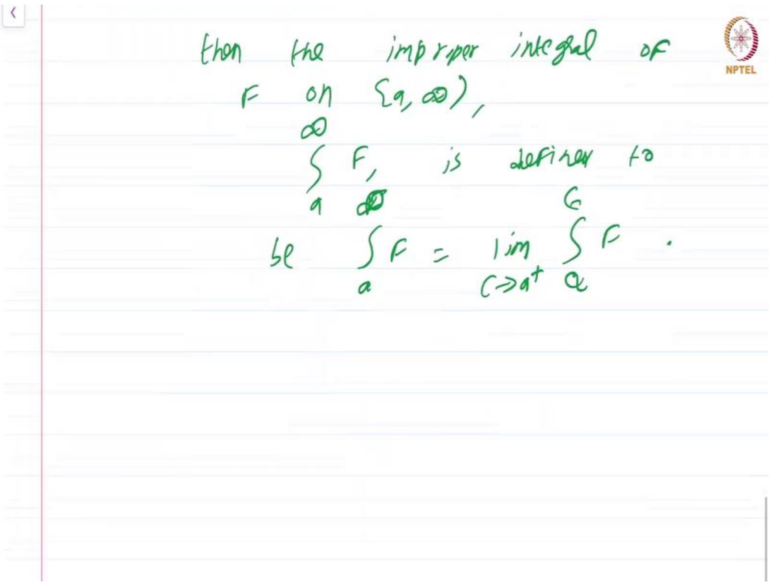

The limit either converges to a finite quantity or it goes to $\pm\infty$ if this happens, then the improper integral of f on $[a, \infty)$ and again this is denoted by $\int_a^\infty f$ is defined to be. Integral $\int_a^\infty f = \lim_{c \to \infty} \int_a^c f$ ok.

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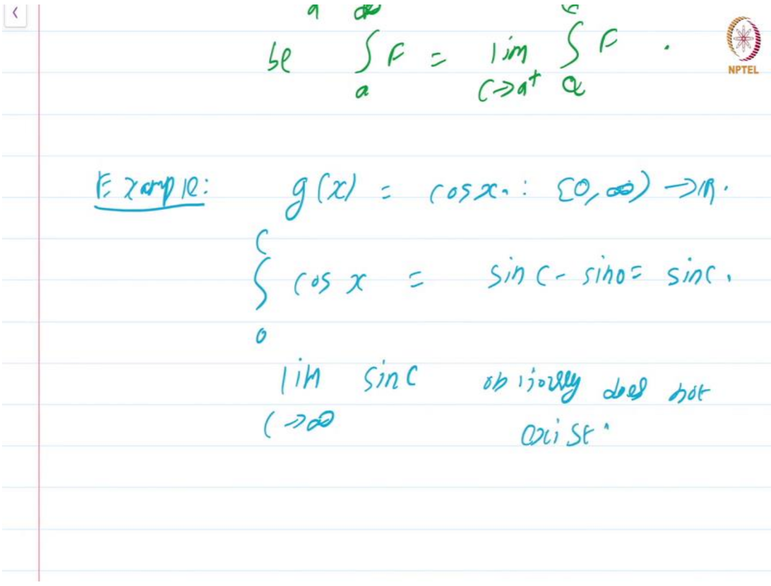

Now, let us see an example. There is nothing really great about this definition, but let us see. Consider the function $g(x) = \cos x$; c. And we are going to define this on $[0, \infty)$ to R. This is a bounded function remember that ok. Of course, $\int_0^{\infty} \cos x = \sin c - \sin 0 = \sin c$ ok.

So, the function $\cos x$ is integrable for any closed interval of the form $[0, c]$ that is straightforward and obvious, but $\lim_{c \to \infty} \sin c$, obviously, does not exist; obviously, does not exist ok.

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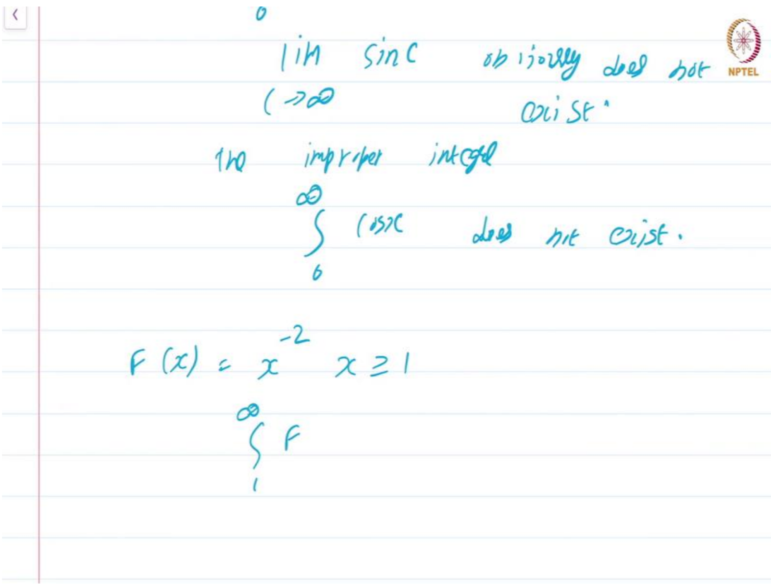

Therefore, the improper integral $\int_0^\infty \cos x$ does not exist ok. So, we cannot assign a sensible meaning to $\int_0^\infty \cos x$. Again, this is expected sort of from the graph. If you think about the graph of $\cos x$, it will be clear to you that one cannot actually expect so, this function to be integrable in any sense.

Let us see another example. Consider $f(x) = x^{-2}\ for\ x \geq 1$ ok. Now, let us try to compute $\int_1^\infty f$ ok.

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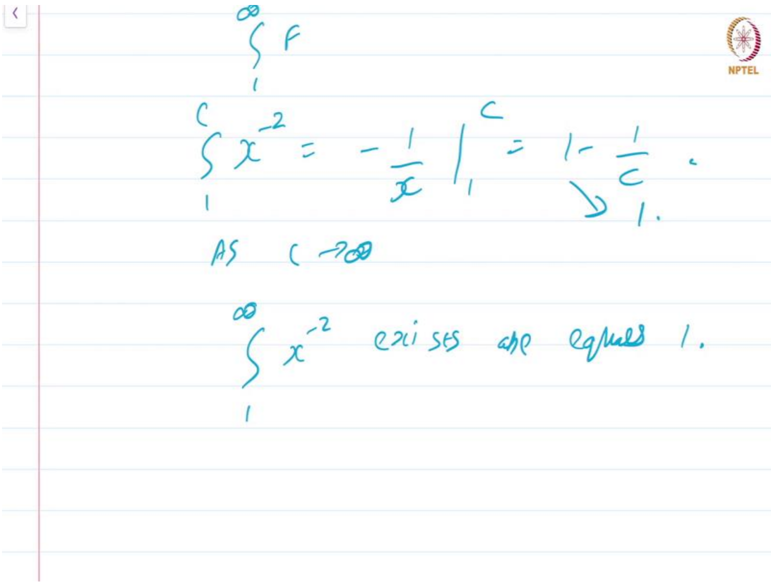

Now, let us take $\int_1^c x^{-2}$and again from basic calculus, this is just $\frac{-1}{x}$ and the limits are 1 and c which is just $1 - \frac{1}{c}$ ok. Now, as c approaches infinity, this above quantity converges to 1 which means integral $\int_1^\infty x^{-2}$ exists and equals 1 ok.

So, the best way to make sense of these improper integrals is to actually draw the graphs of the various functions that we have considered and to see which type of functions seem to have improper integrals and which type of functions do not seem to have improper integrals.

In the next module, we will apply this basic theory to prove a nice test for convergence called the integral test. This is a course on Real Analysis, and you have just watched the module on Improper Integrals.