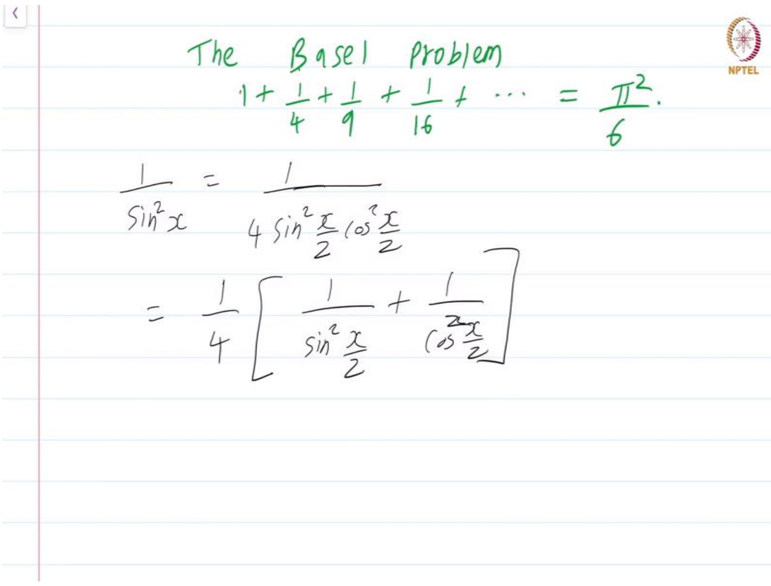
Real Analysis - I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 33.1
The Basel Problem

(Refer Slide Time: 00:20)



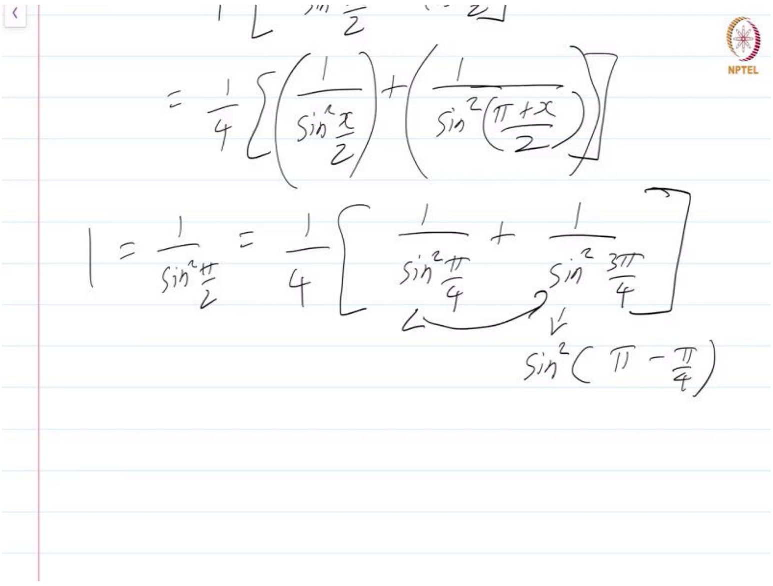
The Basel Problem - Long ago I had promised that we will prove that the sum $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\frac{\pi^2}{6}$, we have already shown in the chapter on sequences and series that this series converges, but we do not know exactly as of yet what it converges to. Now there is an interesting story behind why this is called the Basel problem.

Basel is a place in Switzerland. It is a city and it is a home town of Euler who finally, gave the solution to this problem. Later Euler generalized his methods. His methods were originally not even fully rigorous, but the gaps in his proofs were later fixed. He has generalized this identity to a much more general one involving what is known as the Riemann zeta function.

And his work essentially has consequences for the structure of primes which was taken up by Riemann as the name Riemann Zeta suggests and it is one of the most important things that is studied in Analytic Number Theory. Anyway we will not give Euler's proof. We will give a very modern proof that uses Tannery's theorem. So, we want to show that sum $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\frac{\pi^2}{6}$

I am going to start with a seemingly innocent trigonometric identity and get this. What I am going to do is, I am going to start by saying $\frac{1}{\sin^2 x}=\frac{1}{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}$, this is just the half angle formulas that you are familiar with from high school and I just use the fact that $\sin^2\theta+\cos^2\theta=1$, to rewrite this as $\frac{1}{4}\left[\frac{1}{\sin^2\frac{x}{2}}+\frac{1}{\cos^2\frac{x}{2}}\right]$, ok.

(Refer Slide Time: 02:25)



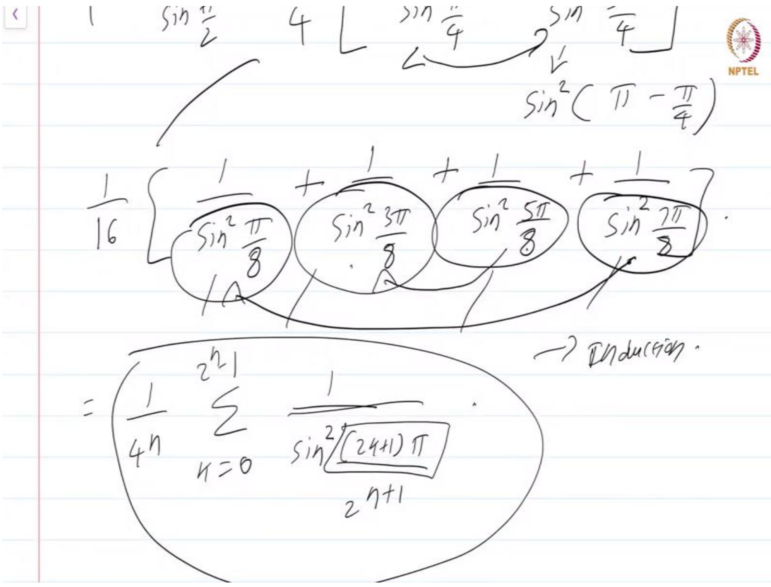
Now what we do is, we rewrite this as $\frac{1}{4}\left[\frac{1}{\sin^2\frac{x}{2}}+\frac{1}{\sin^2(\frac{x+\pi}{2})}\right]$. Again I am just using some famous trigonometric identities that you are familiar with, ok. So, we have derived this expression. Now what we are going to do is, we have landed up with $\frac{1}{\sin^2 x}$.

Something we are going to repeatedly apply the identity that we have that $\frac{1}{\sin^2 x}=\frac{1}{4}\left[\frac{1}{\sin^2\frac{x}{2}}+\frac{1}{\sin^2(\frac{x+\pi}{2})}\right]$. We are going to repeatedly apply this to the point $\frac{1}{sin^2\frac{\pi}{2}}$ which as we all know is just 1 ok.

So, at the first stage we get $\frac{1}{4}\left[\frac{1}{\sin^2\frac{\pi}{4}}+\frac{1}{\sin^2\frac{3\pi}{4}}\right]$, ok. Note this observation will become important in the future. Note that the term here is actually just $\sin^2\left(\pi-\frac{\pi}{4}\right)$, right and we all know that $\sin(\pi - x) = \sin x$.

So, these two terms are actually equal, these two terms are actually equal and that is going to play a role now. Now, what we do is each one of these terms $\frac{1}{\sin^2\frac{\pi}{4}}$ and $\frac{1}{\sin^2\frac{3\pi}{4}}$, we are going to expand again by using the same identity.

(Refer Slide Time: 04:09)



So, you will get another $\frac{1}{4}$. So, which I will write it as $\frac{1}{16}\left[\frac{1}{\sin^2\frac{\pi}{8}}+\frac{1}{\sin^2\frac{3\pi}{8}}+\frac{1}{\sin^2\frac{5\pi}{8}}+\frac{1}{\sin^2\frac{7\pi}{8}}\right]$, let me deal with this $\frac{1}{\sin^2\frac{3\pi}{4}}$ term first. So, I will get $\frac{1}{\sin^2\frac{3\pi}{8}}$, then I have two terms coming from the second part of this identity.

So what will happen is, I will get $\frac{1}{\sin^2\frac{5\pi}{8}}+\frac{1}{\sin^2\frac{7\pi}{8}}$. These are the other two terms ok. So, essentially I am just applying this identity again and to both these terms. So, you will get a $\pi+\frac{x}{2}$. That is why you are getting a $\frac{\pi+\frac{\pi}{4}}{2}$ which is $\frac{5\pi}{8}$ and similarly. Sorry I got a bit ahead of myself.

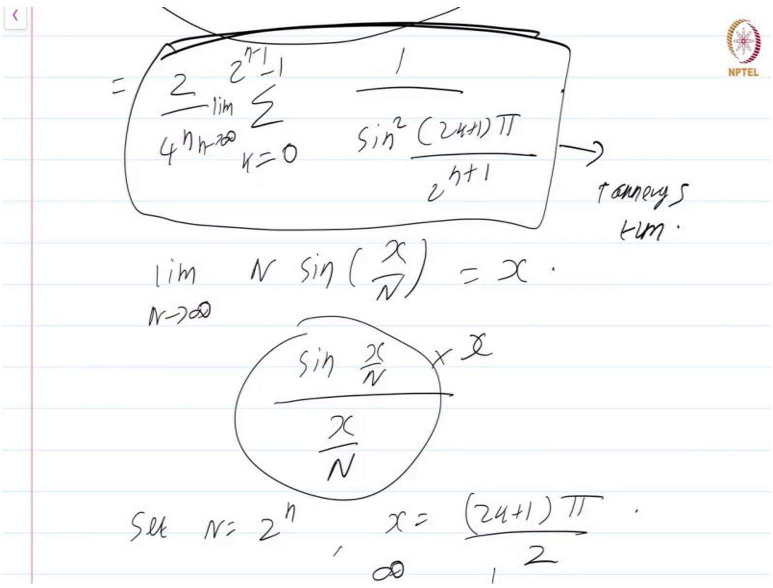
You get $\frac{\pi+\frac{3\pi}{4}}{2}$ which is going to get you $\frac{7\pi}{8}$, ok. So, I hope I said that correctly. Now, again observe that this term is equal to this term because this is just $\sin\left(\pi-\frac{\pi}{8}\right)$ or rather $\sin^2\left(\pi-\frac{\pi}{8}\right)$. Similarly this term and this term are equal, ok.

Now again each one of these terms I expand once again by using the same fact. So, $\sin^2\left(\pi-\frac{\pi}{8}\right)$ will just become $\sin^2\frac{\pi}{16}$, then I will get a $\sin^2(\frac{\pi}{\frac{8+\pi}{2}})$ so on and so forth. I will get many many terms and a bit of thought will tell you that the general form of this is just $\frac{1}{4^n}\sum_{k=0}^{2^n-1}\frac{1}{\sin^2\frac{(2k+1\}\pi}{2^{n+1}}}$, ok.

So, if you want to prove this rigorously just do induction. Induction proves this rigorously. Alternatively just observe that each one of these terms will give two children and each one of these terms will give 2 children, 2 children, 2 children. So, you will get 8 terms and you will get all the odd multiples of π^{2n+1} at the 2^{n+1} stage that is what these are, ok.

So, you can if you are really pedantic, you can just do induction. If not you can just observe that this is in fact true. Now remember this observation that I said the first term is equal to the third term, the second term is equal to the fourth term. Well this holds even in this general expression. So, what you need to essentially observe is that you do not need to sum up till 2^{n-1}. You can stop at 2^{n-1}-1. and just take a 2 outside.

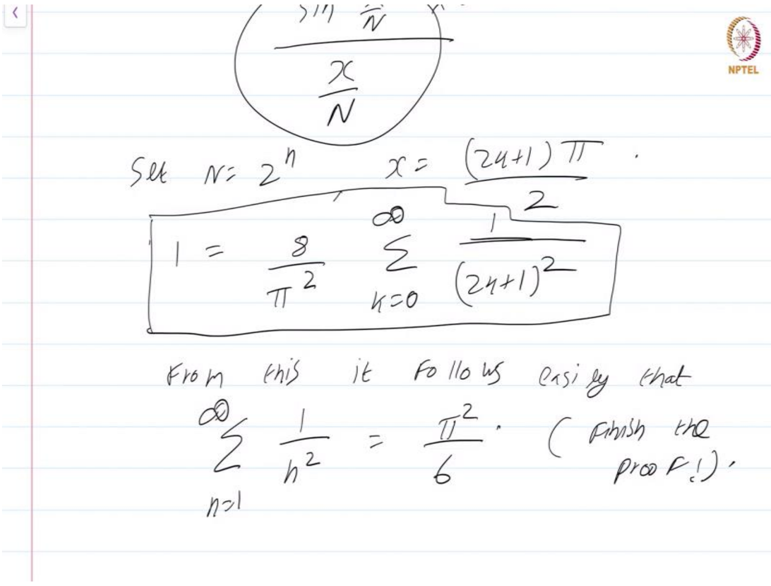
(Refer Slide Time: 07:14)



So, this is just $\frac{2}{4^n}\sum_{k=0}^{2^{n-1}-1}\frac{1}{\sin^2\frac{(2k+1)\pi}{2^{n+1}}}$ ok. So, what I am essentially doing is since I know that there are certain terms that are going to be equal, I am just counting each one of those terms twice. I am just putting a 2 outside and going only till 2^{n-1}, ok.

Again, this will follow just with the moments thought of why this is true. Now here is the part where I am going to apply Tannery's theorem. First observe that $\lim_{N\to\infty} N\sin\left(\frac{x}{N}\right)$, ok. This is just x, this is just x which you can just see by rewriting this as $\frac{\sin\left(\frac{x}{N}\right)}{\frac{x}{N}}$ n goes to infinity, this becomes of the form sin 0 by I mean $\frac{\sin x}{x}$ or$\frac{\sin y}{y}$as y goes to 0. So, this converges to 1 and you are left with an x, ok.

(Refer Slide Time: 08:31)



So, now what you do is, you set $N = 2^n$ and $x = \frac{(2k+1)\pi}{2}$, and you apply this as you take n to infinity in this expression, ok. Just take n to infinity in this expression and do term by term limits and see what happens, ok.

So, when you do that you will get 1 which is what we started off with. We just started expanding $1 = \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$, ok. So, what I have essentially done is, I have gotten a series. I have got that this series sums up to 1 ok, then I take n to infinity that expression will still be valid because the finite sums are all 1. Therefore, when I substitute when I take it all the way to infinity, I will still have this ok.

Then what I am doing is, I am taking the limits essentially inside instead of I want to do limit n going to infinity here ok. I am just taking the limit n going to infinity inside and using this observation that $\lim_{N \to \infty} N \sin\left(\frac{x}{N}\right) = x$, where $N = 2^n$ and $x = \frac{(2k+1)\pi}{2}$. Under these settings I am just taking the limit inside and this part is where Tannery's theorem is used, ok.

The fact that I can take this limit n going to infinity inside is justified by Tannery's theorem. Please check I am being a bit fast simply because this is the end of the course. You should be really good at such manipulations. So, take limit n going to infinity inside take the limits and this is valid because of Tannery's theorem. So, it is your job to check that the hypotheses in Tannery's theorem are all satisfied in our situation.

So, ultimately when you do all these manipulations which is just going to be a few minutes of work, you will get the identity. So, to be extra careful I will draw this box correctly. So, we do not have a perfect rectangle, but no issues. So, we have this identity ok. Now from this it follows easily that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. So, finish the proof, finish the proof ok.

So, this is a very clever and somewhat difficult proof simply because it is not clear exactly how such a proof was thought up of ah. This is from a paper and the author has not mentioned exactly how he has managed to come up with this proof, but nevertheless this is one of the simplest proofs of this identity that I could find.

Of course, the one key fact that makes this proof a non-elementary proof is that you need to know Tannery's theorem, for to see this proof you need to know Tannery's theorem, but we have already done Tannery's theorem in this course. So, I thought why not present a proof that uses stuff that we have already studied ok. So, this finishes the proof.

This is a course on Real Analysis and you have just watched the module on The Basel Problem.