Real Analysis - I
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Lecture – 32.3
The Graphs of \sin and \cos

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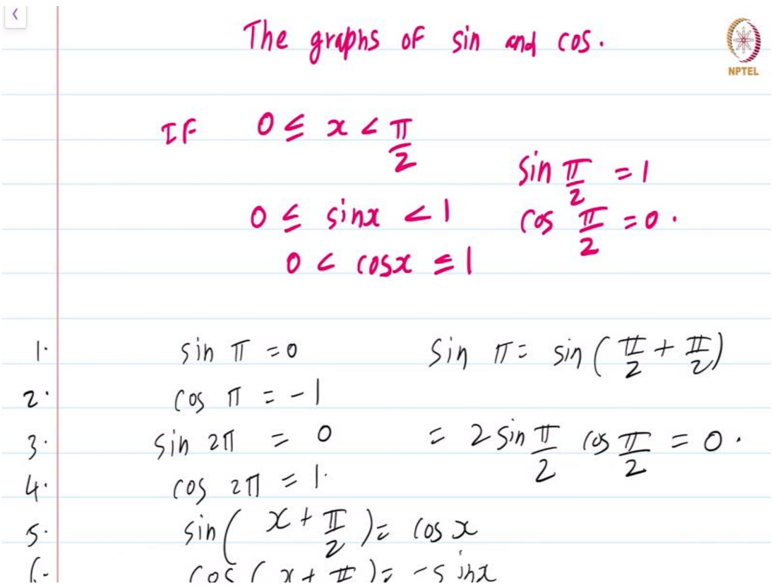

Let us recap what we have achieved so far. We were discus\sing trigonometric functions and π. We have so far shown that if $0 \leq x < \frac{\pi}{2}$ then $0 \leq \sin x < 1$, and $0 < \cos x \leq 1$ok. And we also have $\sin\frac{\pi}{2} = 1$, and $\cos\frac{\pi}{2} = 0$. We have these identities.

Furthermore, we have also proved identities for $\sin(a + b)$ and $\cos(a + b)$ – the addition formula. It is immediate by applying the addition formula, we get additional identities. These identities let me just list a few of them, and you can spend a nice afternoon proving all of these. 1. $\sin \pi = 0$, $\cos \pi = -1$; we have $\sin 2\pi = 0$, and we have $\cos 2\pi = 1$ ok.

Let us say you want to prove one of these, well you just look at $\sin \pi = \sin\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$ which by the addition formula will immediately give $2 \sin\frac{\pi}{2} \cos\frac{\pi}{2} = 0$ ok.

And you can prove all of these identities in quite a similar manner; these are just elementary stuff which you have no doubt seen in your high school ok. Also we have some more identities which I am just going to list and leave it for you to leave you to prove it they are all familiar identities that you have seen in a high school and the proofs are exactly the same. So, we have $\sin\left(x + \frac{\pi}{2}\right) = \cos x$.

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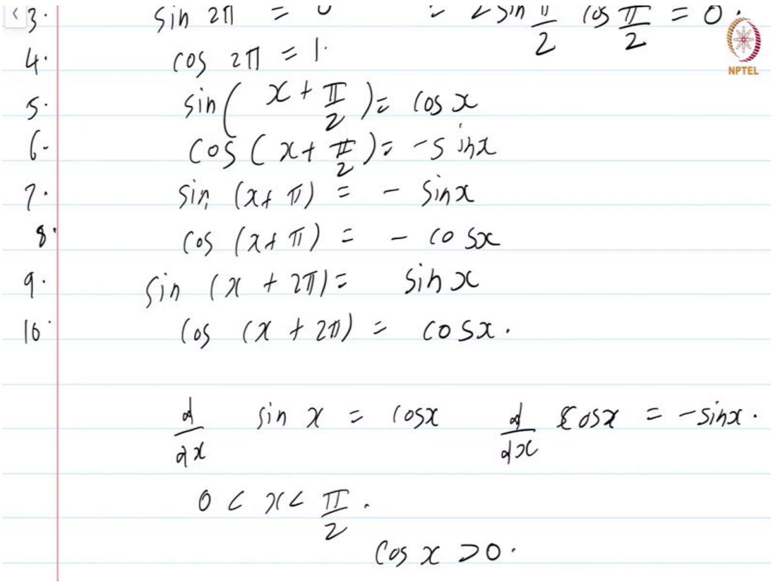

We have $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$, we have $\sin(x + \pi) = -\sin x$, we have $\cos(x + \pi) = -\cos x$, and we have $\sin(x + 2\pi) = \sin x$. And finally, we have $\cos(x + 2\pi) = \cos x$. Now, these are various trigonometric identities all of which follow immediately from what we have established and the addition formulas that is fairly straightforward to see ok.

Now, we are interested in the shape of sin and cos. Let us see how we can derive the shapes roughly of sin and cos using what we have derived. Well, we already know that $\frac{d}{dx}\sin x = \cos x$, and $\frac{d}{dx} = \cos x = -\sin x$ ok. Now, let us focus on the interval $0 < x < \frac{\pi}{2}$; in this interval $\cos x > 0$ right.

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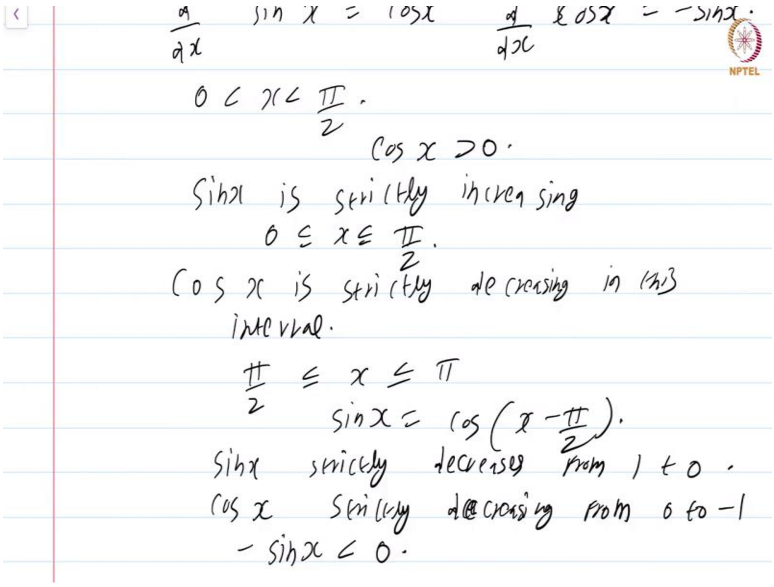

Therefore, what we can conclude is that $\sin x$ is strictly increasing in the interval $0 < x < \frac{\pi}{2}$ simply because the derivative is greater than 0. Exactly in the same way we get that $\cos x$ is strictly decreasing in this interval ok. So, we know that in 0 to $\frac{\pi}{2}$, $\sin x$ is strictly increasing from 0 to 1.

And in a similar way $\cos x$ is strictly decreasing from 1 to 0 ok. Now, we can repeat using some of these formulas involving $\sin\left(x + \frac{\pi}{2}\right)$ and so on. We know that if you take $\frac{\pi}{2} \leq x \leq \pi$, we have not shown you will show that $\sin x = \cos\left(x - \frac{\pi}{2}\right)$, ok.

So, that means, in this interval $\sin x$ strictly decreases from 1 to 0 ok. Whereas, $\cos x$ is strictly decreasing from 0 to - 1, because the derivative is $-\sin x$ which will be less than 0 in this interval which is what we have established right now ok.

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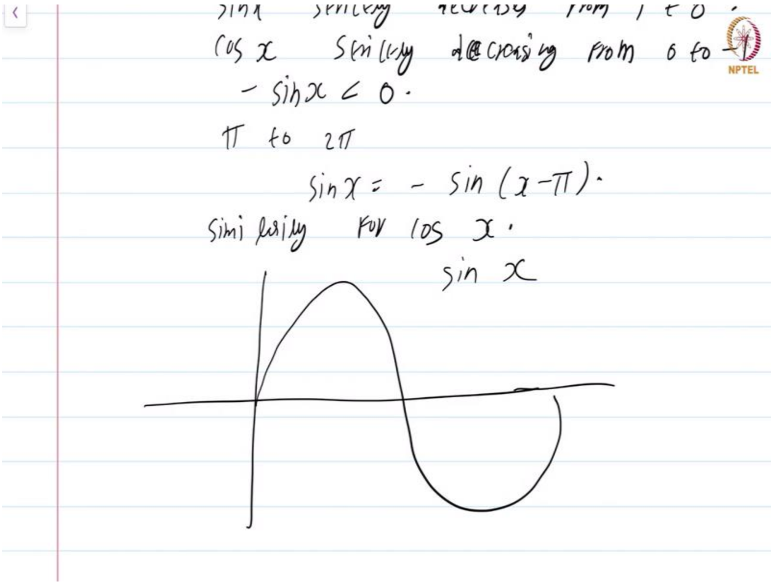

So, similarly for the interval π to 2π we just use the relationship that $\sin x = -\sin(x - \pi)$. And with this we can derive rough pictures of the $\sin$ function, and similarly for cosine, similarly for $\cos x$ ok. So, what we do is, roughly the picture for sin and cosine is going to look like this. This will be $\sin x$ and this will be $\cos x$ roughly.

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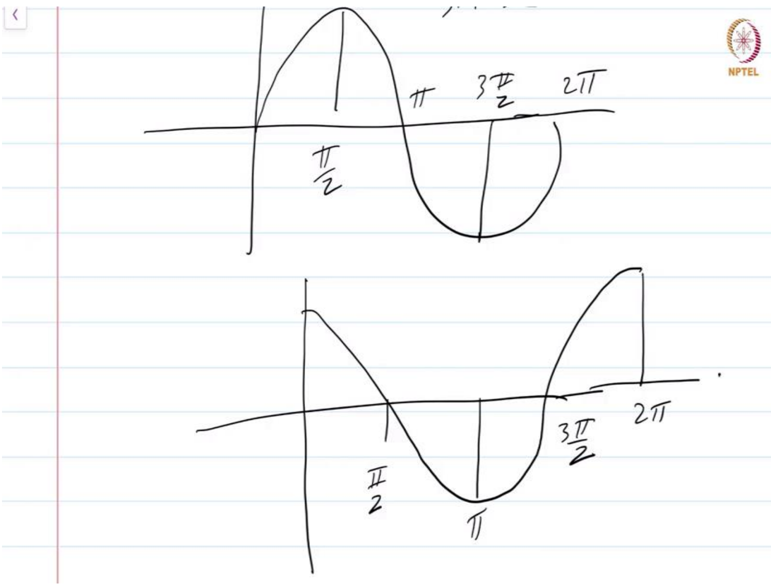

So, this will be $\frac{\pi}{2}$, $\frac{3\pi}{2}$, 2π, and similarly $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π ok. Now, you notice just from the data about the values of sin and cos at these various points and the fact that it is increasing and decreasing I am drawing a picture that looks like this.

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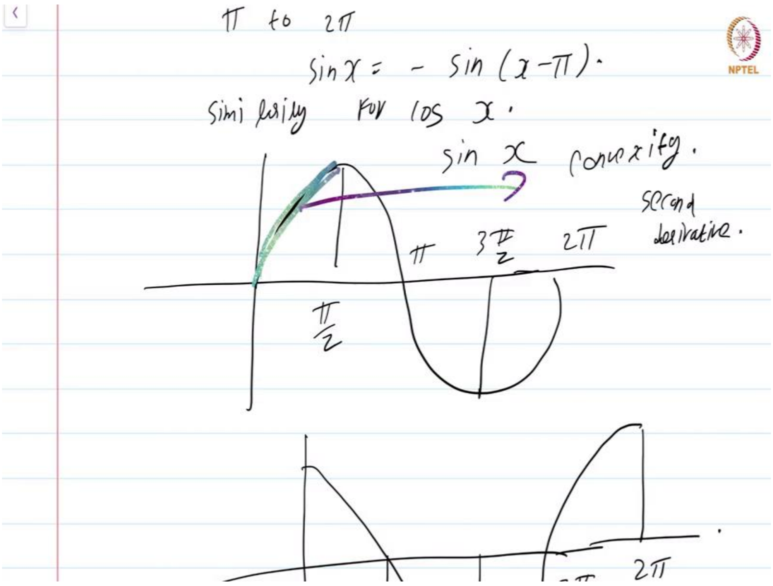

So, let me highlight that in some glitter pen, it looks like this ok, that comes from convexity that comes from convexity. Recall that when we had studied derivatives, we studied convex upwards, convex downwards and so on right. So, the convexity properties are obtained by considering the second derivative ok.

So, what I urge you to do is, compute the second derivatives in each of these intervals and look whether the second derivative is less than 0 or greater than or equal to 0, and try to see that the pictures have to look like the way the curving is going on has to happen in the way that I have described.

So, once you have the values of sin and cosine and whether its decreasing or increasing in the various intervals and also the fact about the convexity, you can get a rough picture of sin and cosine and it is going to look exactly the way you expect it to look.

This is a course on Real Analysis. And you have just watched the module on The Shape of $\sin x$ and $\cos x$.