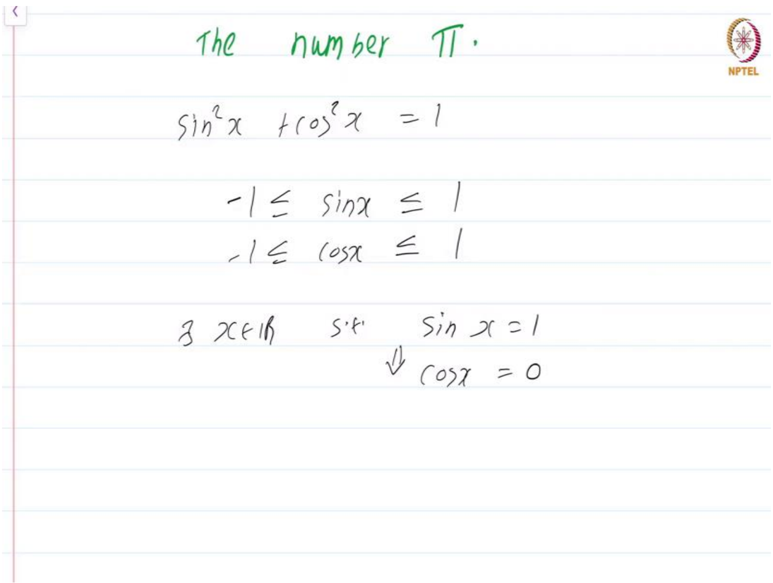
Real Analysis - I

Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 32.2
The Number Pi

(Refer Slide Time: 00:14)

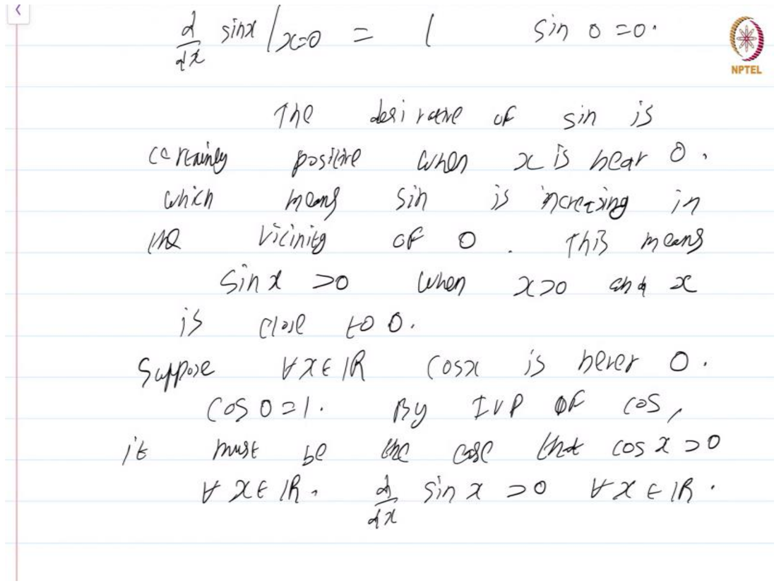


In this module, we are going to define the Number π using the trigonometric functions and calculus. So, we already know that $sin^2x + cos^2x = 1$:this immediately gives us the following inequalities $-1 \leq sinx \leq 1$ and $-1 \leq cosx \leq 1$.

Now, we already know from our familiarity with the trigonometric functions that, all values in between minus 1 and 1 are indeed taken by both sin and cos, but we have not yet proved that rigorously.

So, first let us try to prove that there is a point $x \in R$, such that $sin\, x = 1$. Equivalently because $sin^2x + cos^2x = 1$, we must have $cos\, x = 0$. Both are equivalent $sin\, x = 1$ if and only if $cos\, x$ equal, this is not exactly $sin\, x$ could be -1. So, if $sin\, x = 1$, you have $cos\, x = 0$ ok. This side implication at least we have.

(Refer Slide Time: 01:27)

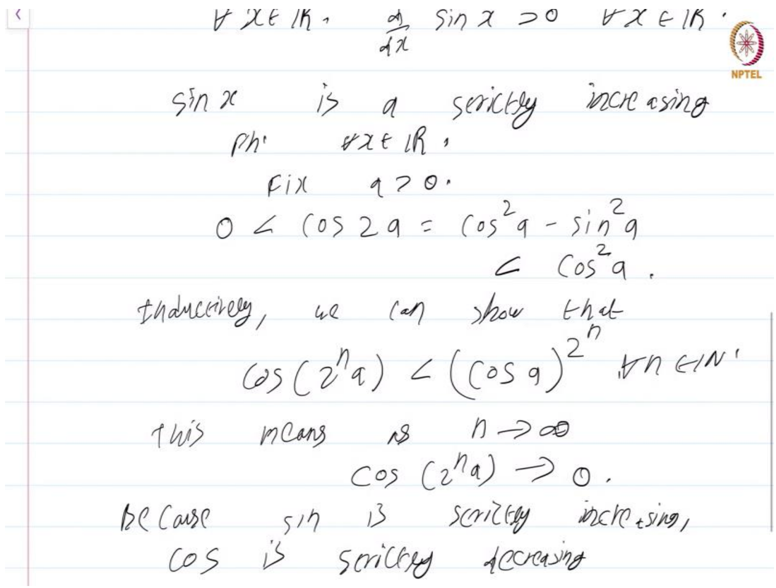


Now, first of all observe that, derivative of $sinx|_{x=0} = 1$. And we also know that $sin\ 0 = 0$; because the derivative of sin is cos and cos is a continuous function. The derivative of sin is certainly positive when x is near 0.

Because the derivative at 0 is 1 and the derivative is $cos\ x$ which is a continuous function; the derivative is positive when x is near 0, which means sin is increasing in the vicinity of 0, ok. So, sin is going to be an increasing function near 0. This means $sin\ x > 0$ when $x > 0$ and x is close to 0, right.

Because $sin\ 0 = 0$, because of that sin will be increasing and therefore, $sin\ x$ will be positive, ok. Now, suppose for all $x \in R$, $cos\ x \neq 0$. We also know that $cos\ 0 = 1$ and $cos\ x \neq 0$ and cos is continuous; putting all this together by intermediate value property of cos. It must be the case that $cos\ x > 0$ for all $x \in R$, right.

(Refer Slide Time: 03:55)



So, what does this tell us? This tells us that $\frac{d}{dx} sinx > 0$ for all $x \in R$, which just means $sin\, x$ is a strictly increasing function on the whole of R; not just near the vicinity of 0, $sin\, x$ is a strictly increasing function for all $x \in R$, ok. Now, we are going to manipulate sin and cos using these trigonometric identities that we have established in the last module to derive what we need, ok.

So, fix $a > 0$. We know that because cos is positive, $0 < cos\, 2a$; because we have just under the assumption that cos is never 0, cos is always positive, so $0 < cos\, 2a$. But from the trigonometric identities that we saw last time; $cos 2a = cos^2 a - sin^2 a$, ok.

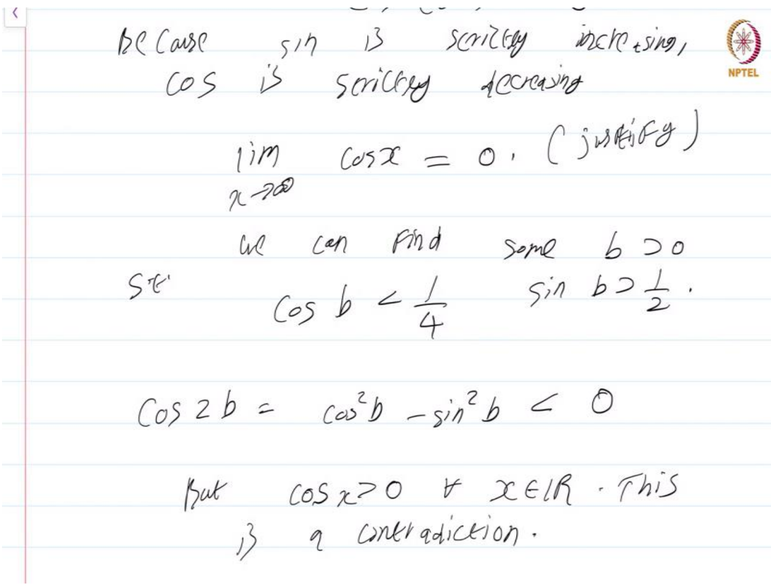
Now, because sin is strictly increasing for all $x > 0$, cos must be strictly decreasing for all $x > 0$. Because $sin^2 x + cos^2 x = 1$, keep that in mind for the moment. We have also seen that if $a > 0$, $sin^2 a$ must be positive; simply because sin is strictly increasing whenever, in fact it is strictly increasing throughout R and $sin\, 0 = 0$. So, $sin^2 a$ is going to be a positive quantity.

And because $sin^2 a$ is positive, this will be strictly greater than $cos^2 a$. Note I am using the fact that $sin\, a \neq 0$ when $a > 0$; because sin is a strictly increasing function and $sin\, 0 = 0$ ok. Sorry, this will be less than cos^a, sorry I completely reverse the inequality that I need, ok.

Now, inductively we can show that $cos(2^n a) < cos(a)^{2^n}$. And this will be true for all n in the natural numbers; just inductively apply the argument that we have given now, ok. This means as n goes to infinity, $cos(2^n a)$ converges to 0, ok. This happens because, we already know that $cos(a) < 1$.

Why do we know that $cos(a) < 1$ when $a > 0$? Because sin is an increasing function, $sin(a)$will be nonzero and positive; so $cos(a)$ cannot be 1 , ok. So, $cos(2^n a)$ approaches 0, ok.

(Refer Slide Time: 07:47)

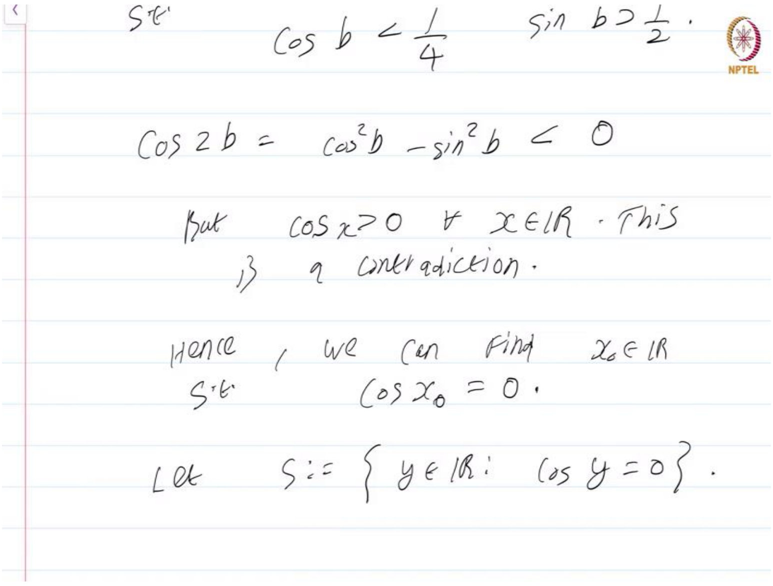


Now, because sin is strictly increasing, because sin is strictly increasing; cos is strictly decreasing. We can use the fact that $cos(2^n a)$ goes to 0 to conclude that $\lim_{x \to \infty} cosx = 0$, ok.

Justify this is fairly easy, because the key step is the fact that cos is strictly decreasing, ok. Now, because $\lim_{x \to \infty} cosx = 0$; we can find some $b > 0$, such that $cosb < \frac{1}{4}$, ok. And, it is clear that, whenever $cosb < \frac{1}{4}$, $sinb > \frac{1}{2}$. In fact, you can get a better bound, but that is all I need; $sin\ b$ is definitely going to be greater than half, ok. How does this help us? Well, again you apply the identity $cos2b = cos^2 b - sin^2 b$.

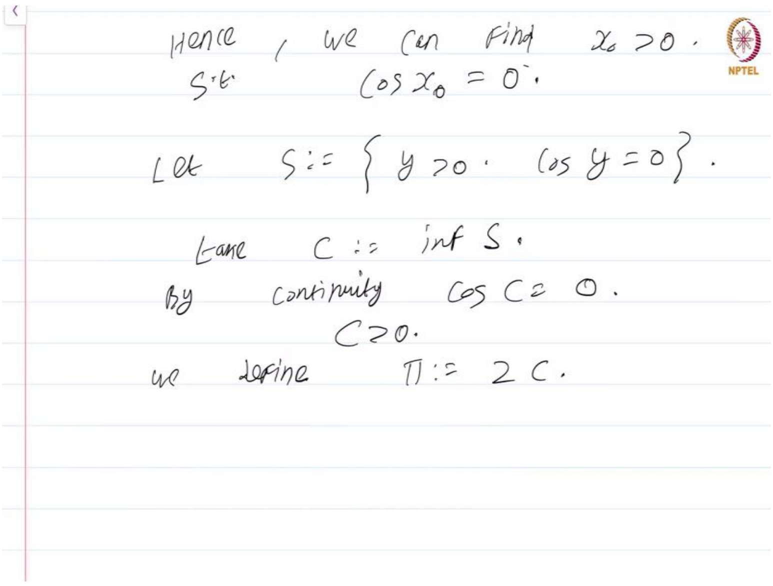
But $cos^2 b - sin^2 b$ when you substitute $\frac{1}{4}$ and $\frac{1}{2}$, this will be less than 0. But $cos > 0$ for all x; that is how we started. This is a contradiction, ok.

(Refer Slide Time: 09:35)



Hence our assumption is wrong; we can find $x_0 \in R$ such that, $cos(x_0) = 0$, ok. Now, what we are going to do now with this is, we can define π the following way. Let S be the set of all $y \in R$, such that $cos\ y = 0$, ok. In fact, all $y > 0$, such that $cos\ y = 0$.

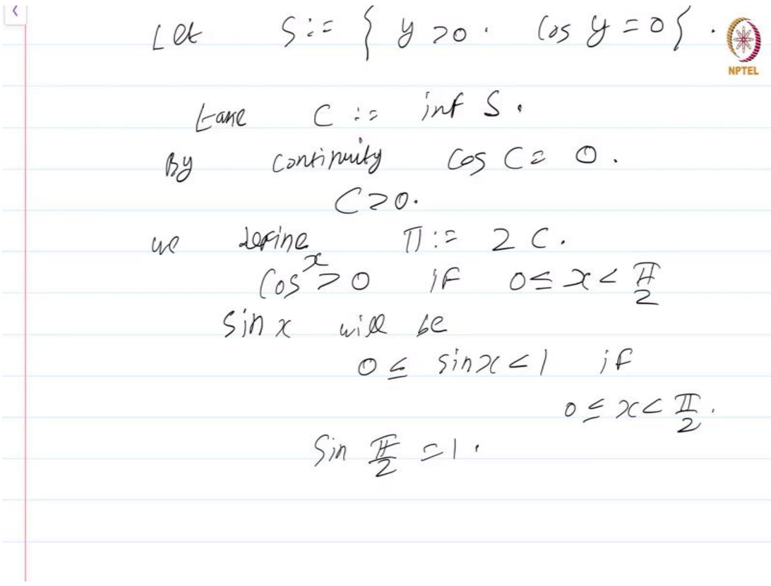
(Refer Slide Time: 10:32)



In fact, we can find $x_0 > 0$, such that this is satisfied. such that $cos(x_0) = 0$, ok. Now, what we are going to do is take $c = \inf S$, ok.

By continuity, $cos\ c\ =\ 0$ ok. And $c\ >\ 0$; because we know that $cos\ 0\ =\ 1$, because of that $c >\ 0$. What we do is; we define c or rather π, we define π to be the quantity by definition as $\pi =\ 2c$. So, essentially what we have found out, this $c = \frac{\pi}{2}$, which we are familiar with.

(Refer Slide Time: 11:46)



And by the way we have defined things, $cos\ >\ 0$, if $0 \leq x < \frac{pi}{2}$; that is simply the way c has been defined, ok. And because of this $sin\ x$ will be $0\ \leq sin\ x <\ \ 1$, if $0 \leq x < \frac{pi}{2}$. And we also know that $sin\left(\frac{\pi}{2}\right) = 1$, ok.

Now, what we are going to do in the next module; now that we have π, we are going to relate π and the trigonometric functions and get various identities such as $sin(x+\left(\frac{\pi}{2}\right)) = cosx$ and $cos(x+\left(\frac{\pi}{2}\right)) = -sinx$ and so on. And using these relations, we are going to somewhat get an approximate graph of the $sine$ and the $cosine$ function.

This is a course on Real Analysis, and you have just watched the module on the Number π.