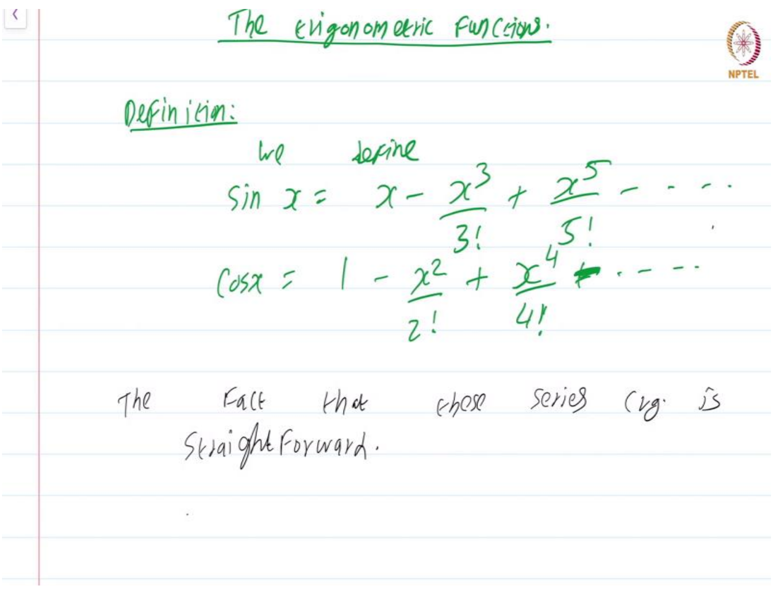
Real Analysis - I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 32.1
Trigonometric functions

(Refer Slide Time: 00:14)

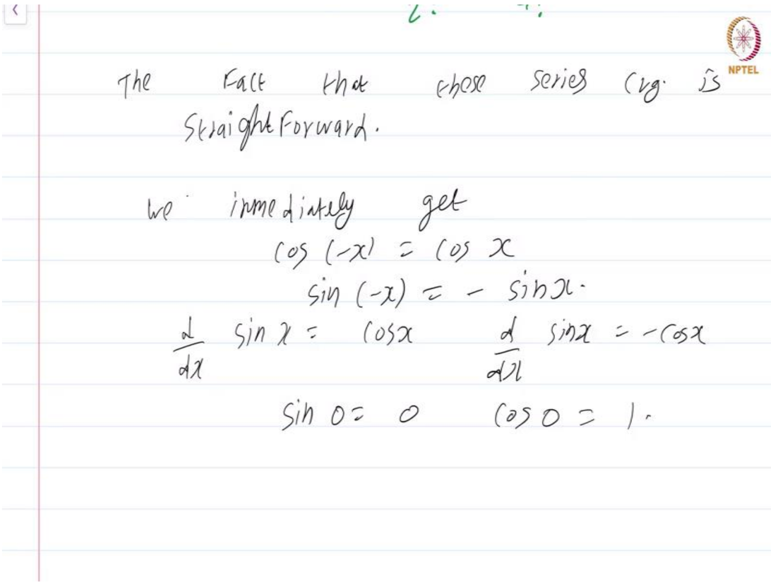


We are now going to define the Trigonometric functions; concentrating mainly on the sin and the cosine and we are going to prove several basic identities that you have already learnt in high school, but this time everything will be done rigorously.

So, first is the definition, we define $sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} -$ and similarly, we define $cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} -$. Again, the signs will change alternatingly ok.

Now, the fact that these two converge is straightforward. In fact, we have seen that the exponential, the series for the exponential converges, you can reuse that to easily show that these two series converge or alternatively, there are several methods. I urge you to think over several proofs of the fact that these two series converge.

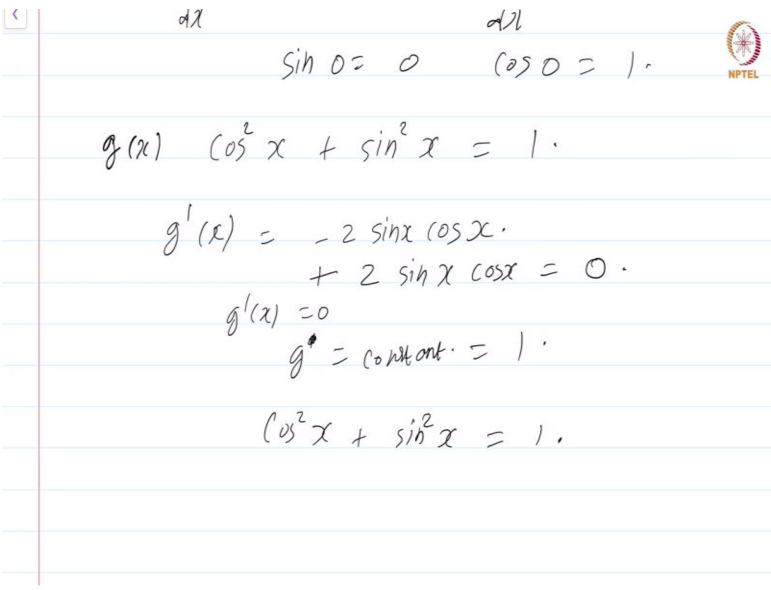
(Refer Slide Time: 01:40)



So, we immediately get some basic properties for free, just from this series expansion. We immediately get $cos(-x) = cos(x)$ and $sin(-x) = -sin(x)$. These two we get for free just by the; just by looking at the series, we also get that $\frac{d}{dx}sin(x) = cos(x)$ and $\frac{d}{dx}cos(x) = -sin(x)$. These two come just by term-by-term differentiation.

So, we have got certain basic properties. We also get before I forget $sin(0) = 0$, $cos(0) = 1$; this also we get for free. So, some of the basic properties of sin and $cosine$ can be immediately read out just by looking at the series. Now, let us go for some deeper properties.

(Refer Slide Time: 02:34)



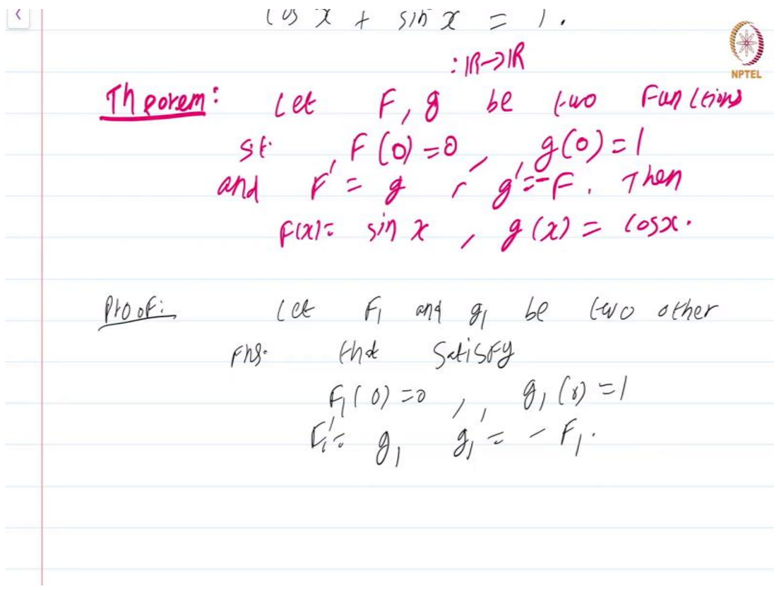
first, the fact that $cos^2x + sin^2x = 1$. Well, how do we get this? Well, we already know that sin and $cosine$ are differentiable and the derivatives of each other are related by the formula $\frac{d}{dx}sin(x) = cos(x)$ and $\frac{d}{dx}cos(x) = -sin(x)$.

So, differentiating this expression $cos^2x + sin^2x$ which we call a function $g(x)$, we get that $g'(x)$ is, derivative of $cos(x)$is $-sin(x)$. So, we get $-2sinxcosx + 2sinxcosx = 0$, ok.

So, we have got that $g'(x) = 0$ that means, g is constant g is a constant function and what is this constant? Well, substitute $x = 0$, we get that this constant is equal to 1. So $cos^2x + sin^2x = 1$ ok. So, we have got even the most famous identity for essentially free.

Now, how do we know that these are the definitions of the power series I wrote down, how do we know that these two must exactly be the sin and $cosine$ function that you are familiar from familiar with from high school?

(Refer Slide Time: 04:08)



Well, we know that because of the following theorem which is very similar in spirit to the earlier theorem which we saw on the exponential function. Let f, g be two functions two functions such that $f(0) = 0, g(0) = 1$ and $f' = g, g' = -f$ ok; just assume this much.

So, we are assuming that f and g are functions from R to R. We are also assuming that f and g are differentiable and the derivative of f is g and the derivative of g is $-f$. How did I forget that and we have $f(0) = 0\ and\ g(0) = 1$.

Then, $f(x) = sinx\ and\ g(x) = cosx$ ok. So, essentially there are unique functions with these properties. Therefore, the sin and cosine function that I have wrote down must agree with the familiar sin and cosine function defined in terms of triangles, that you already saw in school. So, how do we prove this? Well, it is similar in spirit to what we did for exponential; but slightly more involved because there are two functions involved in and they are acting in tandem.

So, what we do is the following. Let f_1 and g_1 be two other functions that satisfy $f_1(0) = 0, g_1(0) = 1, f_1' = g$ and $g_1' = -f_1$. Let us assume that there are another set of functions.

What we will show is $f = f_1$ and $g = g_1$; in particular, taking f_1 and g_1 to be sin and cosine, we get that there is a unique pair of functions that satisfy these properties and that pair is given by $sinx$ and $cosx$ ok. So, now what we have to do is we have to consider a slightly more tricky combination of f and g.

(Refer Slide Time: 06:23)

$$f_1' = g_1 \qquad g_1' = -f_1.$$

$$f \times \quad f g_1 - f_1 g = a f$$

$$g \times \quad f f_1 + g g_1 = b g$$

$$f^2 g_1 - f f_1 g - g f f_1 - g^2 g_1 = af - bg.$$

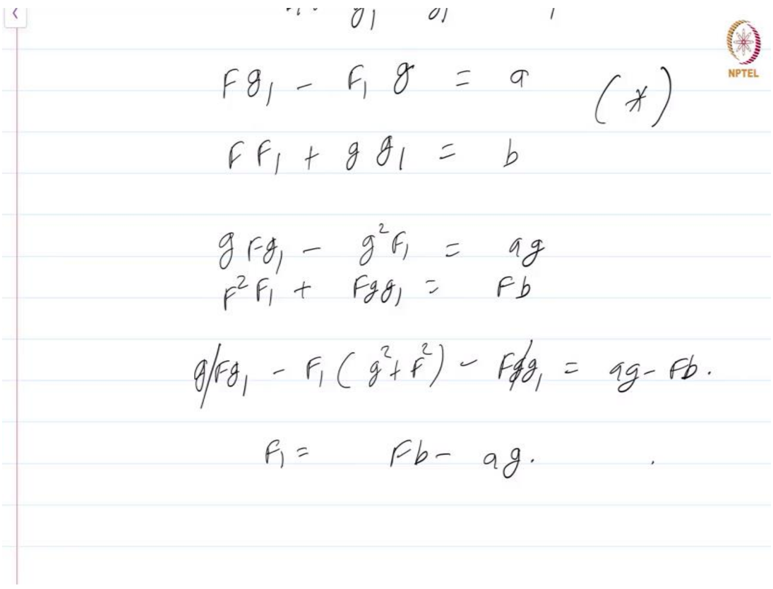
We consider $fg_1 - f_1g$ and we also consider $ff_1 + gg_1$. We consider these two combinations of f, f_1, g, g_1. Now, immediately differentiating in a minute or 2, you will conclude that the derivatives of both these functions $fg_1 - f_1g\ and\ ff_1 + gg_1$, the derivatives are both 0.

So, we must have constant such that this combination is a and this combination is b ok. Now, what we do is we multiply the first equation by f, we multiply the first equation by f and we multiply the second equation by g, we multiply the second equation by g.

So, and then we subtract, we subtract both equations. So, what we will essentially get is we will get $f^2g_1 - ff_1g - gff_1 - g^2g_1$ ok. We will get this combination this is equal to $af - bg$ fine. So, now, what we do is we already know that f^2.

Wait a second; I did the wrong multiplication. I did the wrong multiplication. Just a second. Yeah, these things are a bit of trial and error. So, this is a part of the natural process of learning. What we will do is we will multiply the first equation the first equation by g and the second equation by f.

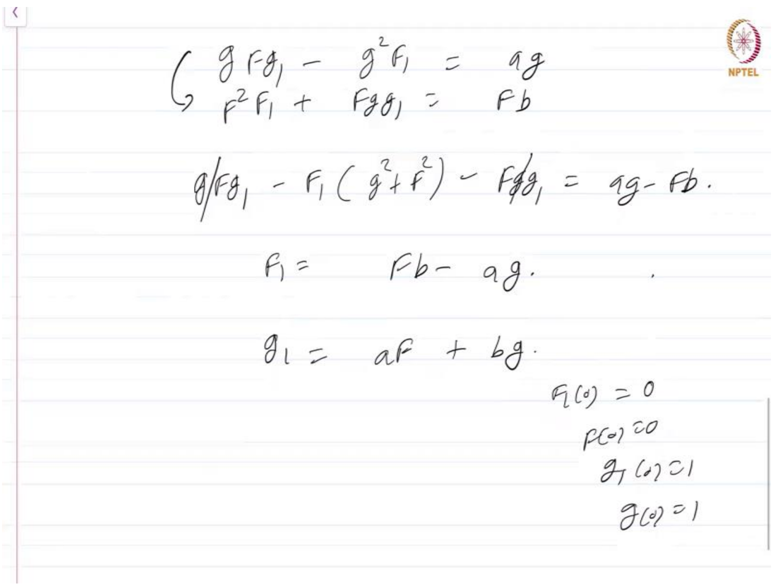
(Refer Slide Time: 08:08)



So, let me rewrite the equations are not butcher what is already there which I am going to now call star; I am going to call this pair of equation star. Now, what I am going to do is I am going to multiply the first by g and the second by f. So, what we will get is $gfg_1 - g^2f_1 = ag$ and we will get $f^2f_1 + fgg_1 = fb$. Now, subtract these two equations, what we will get is $gfg_1 - f_1(g^2 + f^2) - fgg_1 = ag - fb$ f.

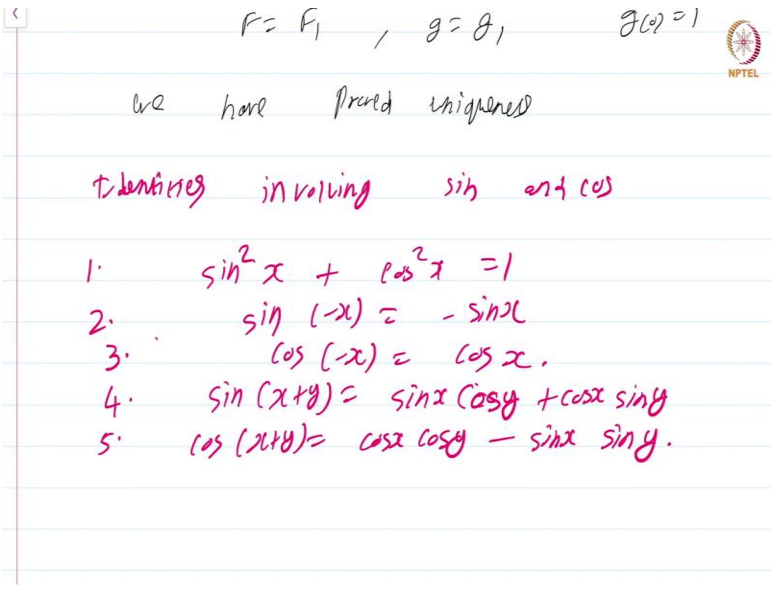
Now, let us see what happens. So, we can now use the fact that $g^2 + f^2 = 1$ and the first and the third terms cancel ok. So, we find end up with $f_1 = fb - ag$ ok. Now, what you do is you look back at the original equations $fg_1 - f_1g = a$ and $ff_1 + gg_1 = b$. Again, multiply by g the first equation and f to get this $gfg_1 - g^2f_1 = ag$.

(Refer Slide Time: 09:48)



And this time, add these two, add these two equations now and a bit of algebraic manipulation will give you $g_1 = af + bg$ ok. So, we have managed to get the expressions for f_1 and g_1 in terms of f and g. Now, this helps us a lot because we know that $f_1(0) = 0$ and $f_1(0)$ =and $g_1(0) = 0$ and $g(0) = 1$. We know all of this ok.

(Refer Slide Time: 10:30)



Now, setting $x = 0$ in both these equations immediately gives $f = f_1$ and $g = g_1$ ok. This comes because you can determine the constants; the constant a will be 0 and the constant b will be 1. So, we have shown in uniqueness. We have proved uniqueness.

Now, interesting fact is that if you ignore the conditions $f_1(0) = 0$ and $g_1(0) = 1$, this double star, this set of equations expression for f_1 and g 1 in terms of f and g is still valid. In addition, if you assume that $f_1(0) = 0$ and $g_1(0) = 1$. We get gives $f = f_1$ and $g = g_1$.

Why am I making this seemingly unnecessary remark? Because using that we can immediately prove the following properties or identities involving sin and cos. Let me list all of them and we will prove some of them the others have already been proved or they are too easy.

So, number 1, which is the one we have been using repeatedly; $sin^2x + cos^2x = 1$, we have already shown this.

Number 2, $sin(-x) = -sin(x)$ and number 3, $cos(-x) = cos(x)$; these two follow immediately from the power series.

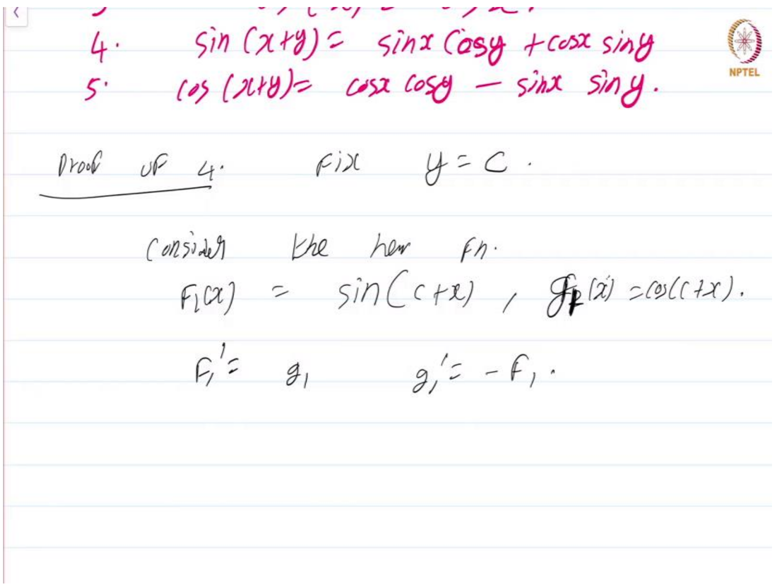
It is an interesting exercise to forget the power series expansion of sin and cos and try to prove this directly from the fact that sin and $cosine$ are a pair of functions that satisfy the derivative of sin is cos and the derivative of cos is $-sin$.

But from the basic properties, you can also prove 2 and 3 ok. 4, is the famous sin rule

$sin(x + y) = sinx\ cosy + cosx\ siny$ and number 5,

$cos(x + y) = cosxcosy + sinx\ siny$, ok. Let me prove 4 and the proof of 5 is entirely similar ok.

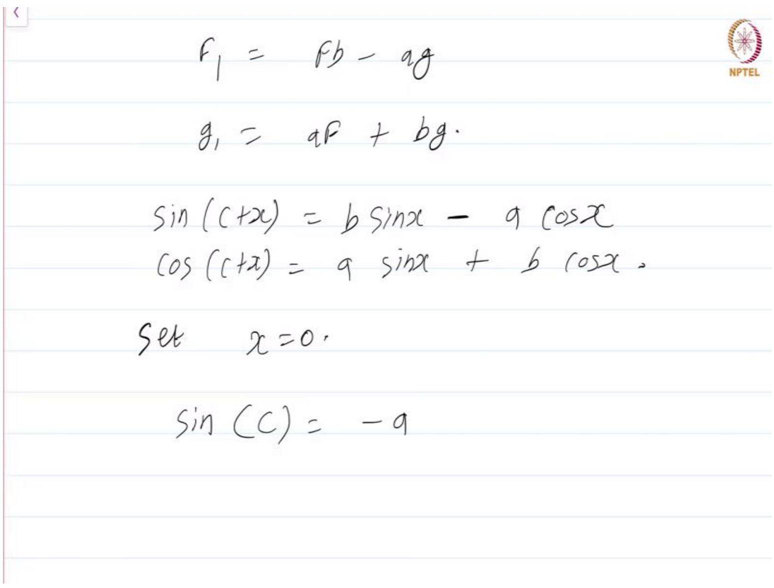
(Refer Slide Time: 12:57)



So, proof of 4. Fix y equal to some constant c and consider the new function the new function $h(x) = sin(c + x)$ or rather let me use reuse notation set $f_1(x) = sin(c + x)$, $g_1(x) = cos(c + x)$, ok.

Now, observe that $f_1' = g_1$ and $g_1' = -f_1$. This follows immediately from the chain rule. How does this help us? Well, the unnecessary remark that I made that this equation double star or this pair of equations double star is still valid just with the assumption that that $f_1' = g_1$ and $g_1' = -f_1$ is very useful now ok.

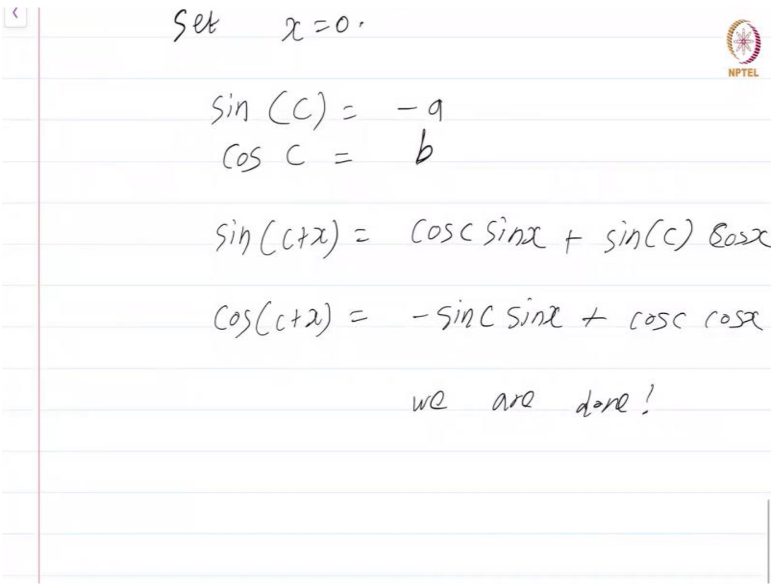
(Refer Slide Time: 14:21)



It is very useful. What does this give us? Well, this gives us f_1 is equal to let me go back because my memory is not what it used to be; $f\ b\ -ag$, $f_1 = fb - ag$ and $g_1 = af + bg$.

So, let me just translate this pair of equations into more concrete stuff, what we actually have is $sin(c + x) = bsinx - acosx$ and $cos(c + x) = asinx + bcosx$ ok. Now, what we do is set $x =\ 0$, what we get is $sin(c) = -a$; $-a = sin(c)$.

(Refer Slide Time: 15:38)



And second equation gives us $cos(c) = b$, just by setting $x\ = 0$. Well, what does this give us? This gives us that $sin(c + x) = cosc\ sinx + sinc\ cosx$ ok. It became plus $sinc$; because $sin(c) = -a$ is minus a and similarly, $cos(c + x) = -sinc\ sinx + cosc\ cosx$ and we are done.

In fact, I said I will prove 4 but we got both 4 and 5 right. We got both for free, excellent. So, we have proved all the basic properties of sin and $cosine$; not all the basic properties, a good chunk of the basic properties of sin and cosine very easily and effortlessly.

So, the months of your life that you spent in high school studying these identities sort of seems like futile because we were able to get it in 20 minutes ok. So, excellent. We will see more properties of sin and $cosine$ in the next module, where we are going to now show how the graph of sin and $cosine$ looks like and to find out how the graph of sin and $cosine$ looks like, we first need to meet one of the most important characters in mathematics the number pi.

This is a course on Real Analysis, and you have just watched the module on the Trigonometric functions.