Real Analysis - I
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Lecture – 31.3
The Logarithm

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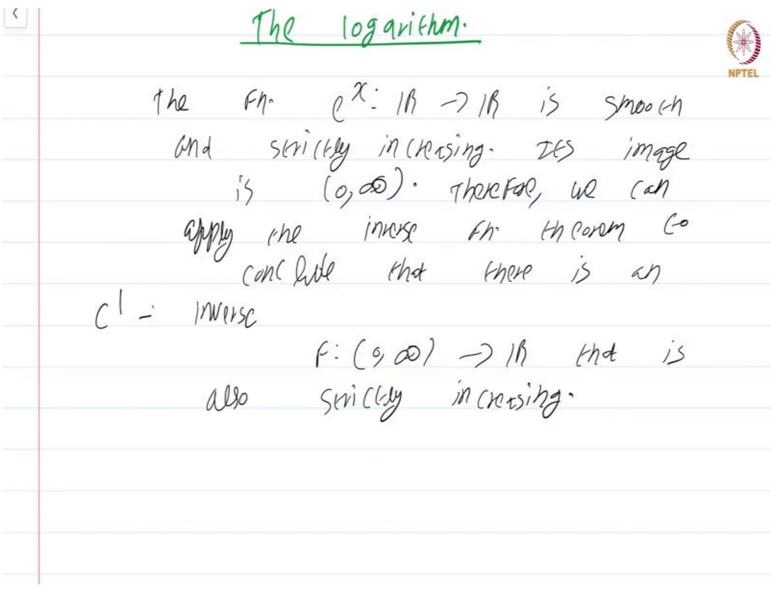

The function $e^x: R \longrightarrow R$ is smooth and strictly increasing. Its image is $(0, \infty)$; we have seen all these aspects in the module on the exponential function. Therefore, we can apply the inverse function theorem to conclude, that there is an inverse, differentiable inverse. In fact, C^1-inverse. from what we have already seen; there is a C^1 inverse, such that or rather C^1 inverse $f: (0, \infty) \longrightarrow R$ that is also strictly increasing. Again every single statement that is made in this paragraph, can be easily justified using the content that is already been developed. So, I am going to leave it to you to check this in its entirety, ok.

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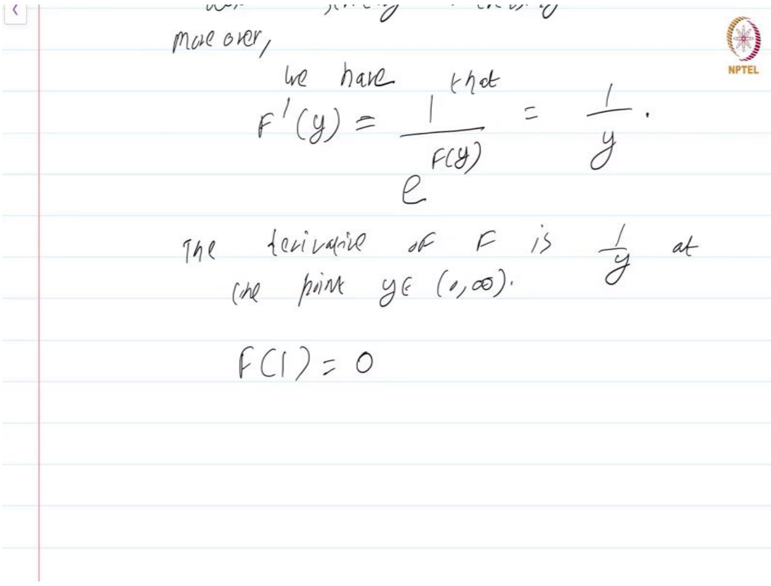

Moreover, we have that, f composed with or rather let me directly compute the derivative, let me directly compute the derivative; $f'(y) = \frac{1}{e^{f(y)}} = \frac{1}{y}$, right. The inverse function theorem tells you how to compute the derivative of a function also, right. So, the derivative at y of the inverse of the exponential function will be nothing but $\frac{1}{e^{f(y)}}$, ok. Now, exponential and this function f are nothing, but inverses.

So, this is nothing but $\frac{1}{y}$, ok. So, we have got that the derivative of this function f is $\frac{1}{y}$ at the point $y \in (0, \infty)$, ok. We also know that this function f must satisfy $f(1) = 0$; simply because exponential at 0 is 1, ok.

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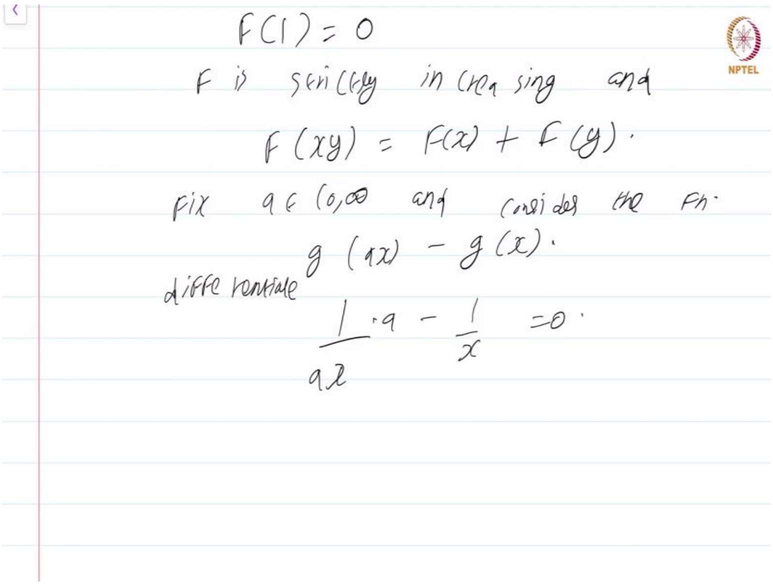

Now, we already know that f is strictly increasing and it also satisfies this nice identity that $f(xy) = f(x) + f(y)$. How do you show this? Well, it is very similar to how we showed a similar property for exponential that, $e^{a+b} = e^a e^b$.

You can use that to prove this or you can prove this directly by considering the function. So, fix a, this proof is going to be very similar; fix $a \in (0, \infty)$ and consider the function $g(ax) - g(x)$. Consider this new function; differentiate this function. Well, by chain rule, you will just get $\frac{1}{ax} a - \frac{1}{x} = 0$.

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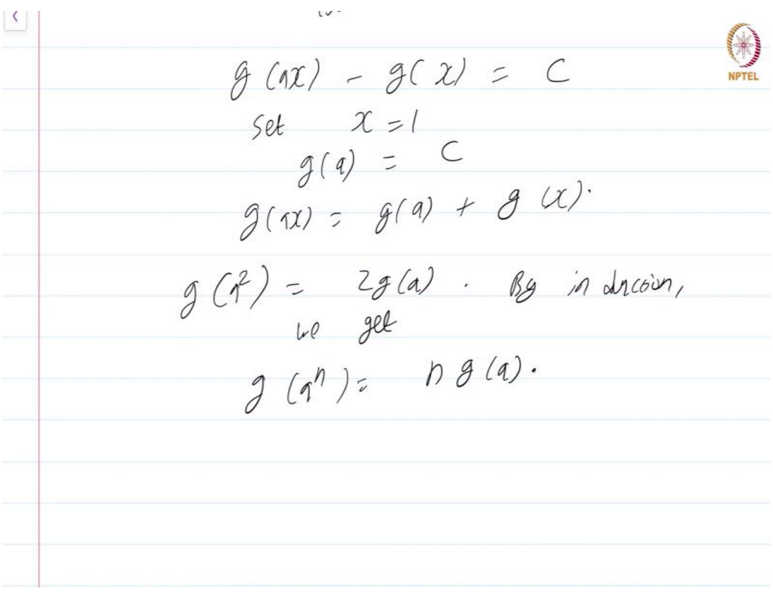

This just means that $g(ax) - g(x) = C$ is just constant, is just some constant. Now, set $x =$; we get $g(a) - g(1) = 0$. So, we get $g(a) = C$ in other words $g(ax) = g(a) + g(x)$. Now, again we can just consider $g(a^2)$ and we will get $g(a^2) = 2g(a)$. And similarly by induction, we get $g(a^n) = ng(a)$., ok.

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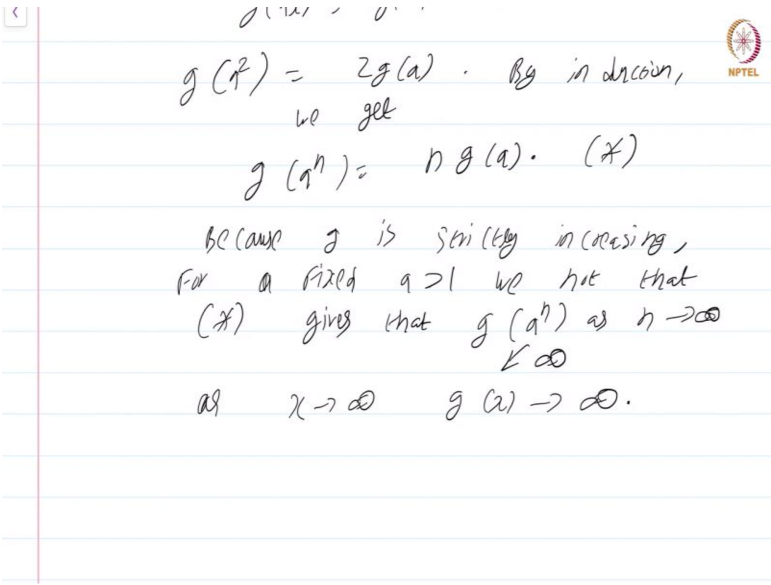

Now, because g is strictly increasing, remember I wrote that back, g is strictly increasing, for a fixed $a > 1$; we note that, star gives $g(a^n)$ as $n \to \infty$ diverges to ∞, right. a^n as you keep boosting a; if $a > 1$, of course $g(a)$ is not going to be 0, because we know that g is strictly increasing and the fact that $g(1) = 0$.

So, as $g(a^n) = ng(a)$, so as n goes to infinity, $g(a^n)$ converges to infinity; but g is strictly increasing, that just means that as x goes to infinity, $g(x)$ converges to infinity, ok. So, this function g is strictly increasing and it goes to infinity as x goes to infinity and at the point 1, it is going to be 0, ok.

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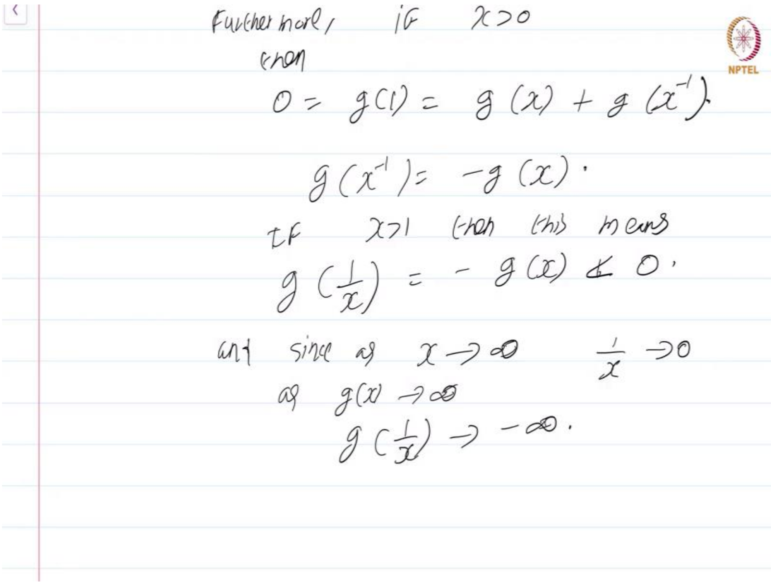

Furthermore, if $x > 0$; then $0 = g(1) = g(x) + g(x^{-1})$ right. This is just coming from the identity that $g(xy) = g(x) + g(y)$, right. So, which means $g(x^{-1}) = -g(x)$; if $x > 1$, then this means $g\left(\frac{1}{x}\right) = -g(x)$, which is going to be negative, ok. And since as $x \to \infty$, $\frac{1}{x} \to 0$ and $g(x) \to \infty$, $g\left(\frac{1}{x}\right) \to -\infty$; because $g\left(\frac{1}{x}\right) = -g(x)$.

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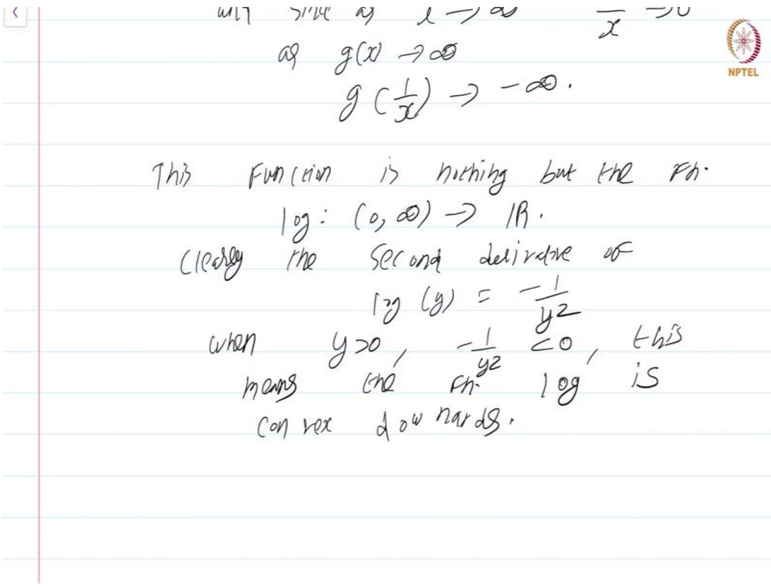

So, this function g is nothing but the function $log: (0, \infty) \longrightarrow R$ ok. Now, we have shown several properties of the logarithm; but we still do not have enough data to draw its graph, for that we need the second derivative. The second derivative of $\log(y) = \frac{-1}{y^2}$; we already know this, because the derivative is $\frac{1}{y}$.

So, this has got to be when $y > 0$, $\frac{-1}{y^2} < 0$; this means, the function log is concave, ok. It is concave, it is not concave sorry; it is not concave, it is convex downwards, it is convex downwards, ok.

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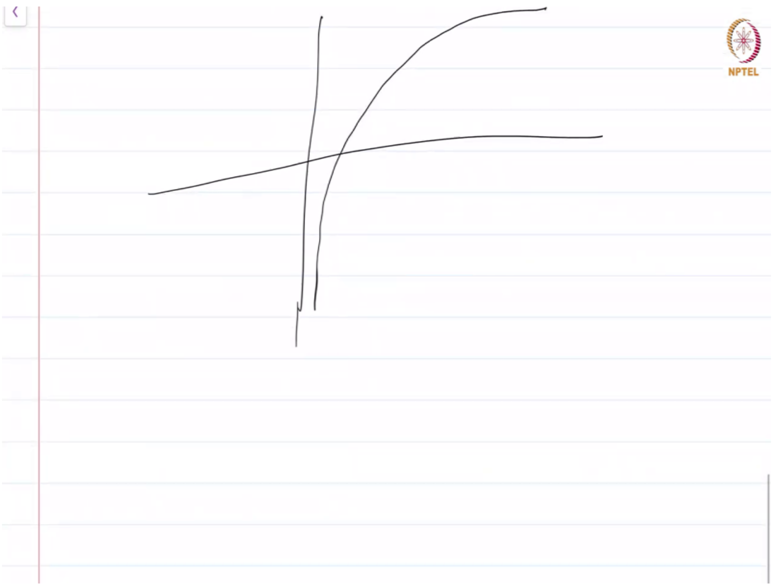

What this means is that, the function essentially looks like this, a familiar picture of the logarithm function, ok. So, we have established in record time, all the basic properties of logarithm essentially covering years of your school syllabus from middle school in a matter of 10 to 15 minutes; let us now proceed to something newer.

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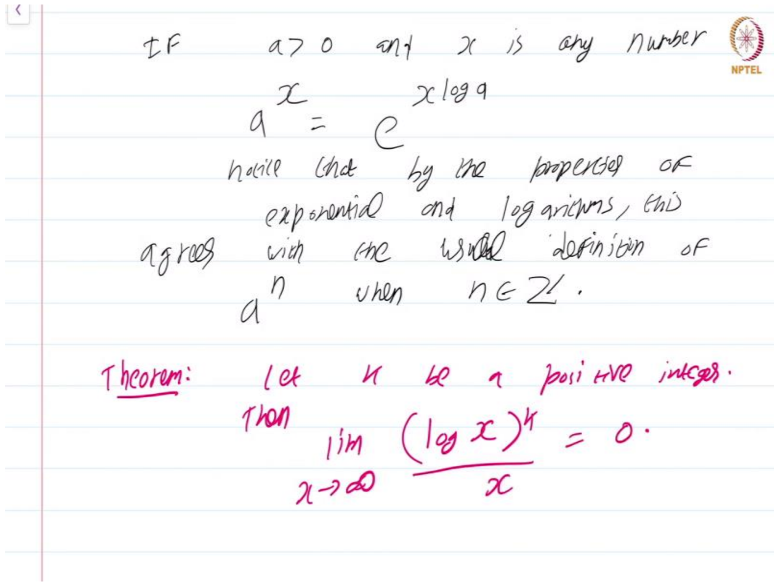

So, if $a > 0$ and x is any number; we can now define a^x to be $e^{\log a}$. This is a long pending definition, we had made this earlier; but without the existence of the exponential and logarithm done in great detail, this definition will not really make sense.

Now, notice that by the properties of exponential and logarithms, this agrees with the usual definition of a^n, when n is coming from the integers, ok. So, this is not something entirely pulled out of the air; this makes, this is a sensible definition, ok.

I am going to just prove one property of the logarithms that is really nice.

Theorem, let k be a positive integer. Then $\lim_{x \to \infty} \frac{(\log x)^k}{x} = 0$. We already know that the exponential function goes to infinity faster than any polynomial; we have encountered this phenomenon several times in this course.

This is saying something interesting, take the logarithm, we know that it goes to infinity; boost the speed at which it goes to infinity by taking power k, it still goes to infinity far slower than even the linear factor just x. So, the logarithm function grows really slowly and that is captured quantitatively and analytically in this theorem.

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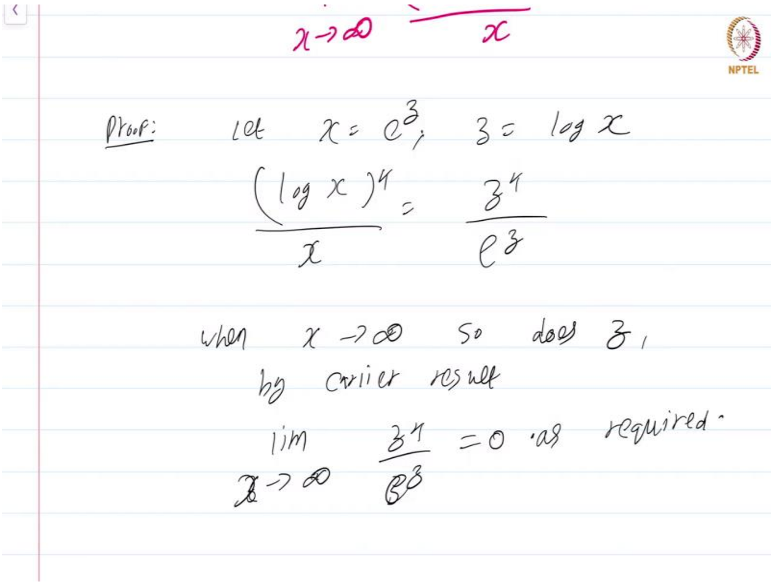

And the proof is fairly easy because we have already done the hard work with using the exponential function. So, what you do is, you set $x = e^z$; in other words $z = \log x$ ok, then $\frac{(\log x)^k}{x} = \frac{z^k}{e^z}$.

Now, when $x \to \infty$, , so does z; but by earlier result limit $z \to \infty$ or rather $\lim_{x \to \infty} \frac{z^k}{e^z} = 0$ as required.

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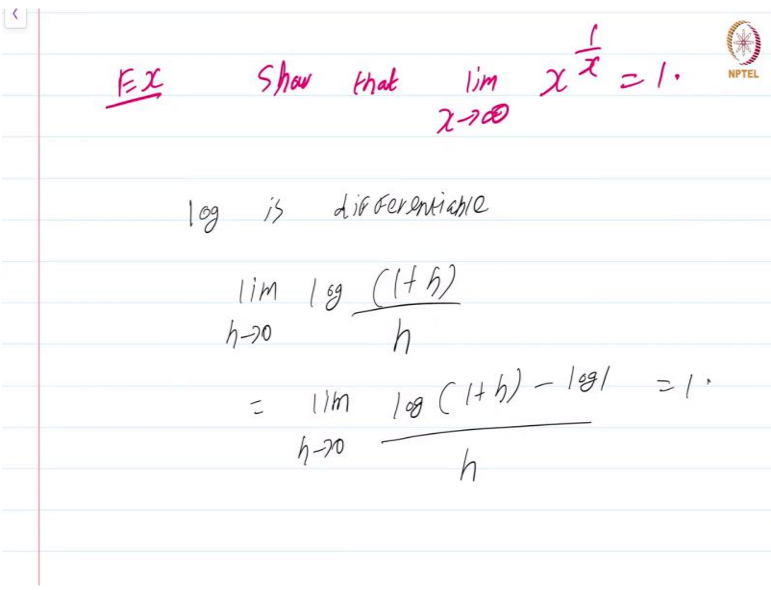

Now, there is an easy corollary of this, which I am going to leave it as an exercise; show that $\lim_{x \to \infty} x^{\frac{1}{x}} = 1$. And recall that we have shown something similar to this for sequences, ok. Now, we end with yet another proof of a famous limit formed for the exponential.

What we do is the following; we know that the function log is differentiable that is how it was constructed as an inverse of the exponential function, we essentially created the log function using the inverse function theorem, therefore it is differentiable.

Therefore, it is differentiable at the point 1; the derivative is given by the limit,

$\lim_{h \to 0} \frac{\log(1+h) - \log 1}{h} = 1$. We know this because the derivative of log is $\frac{1}{x}$ or $\frac{1}{y}$, and y in this case is 1, ok.

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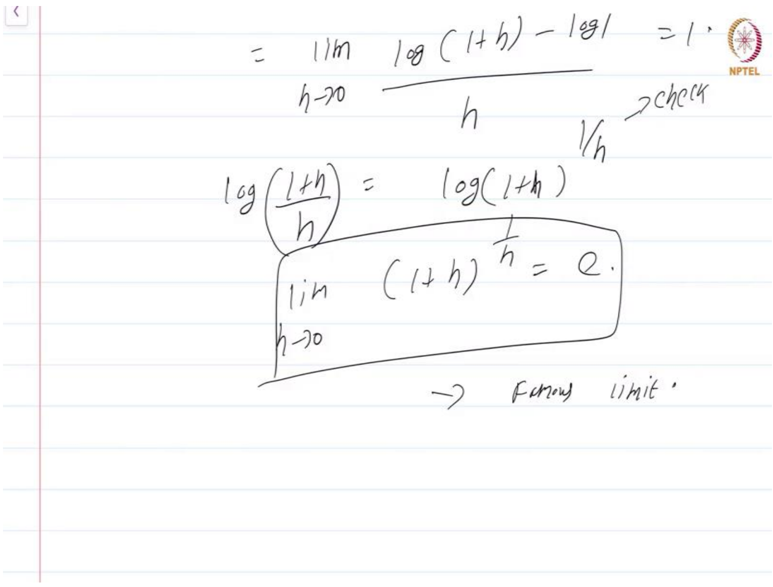

Now, $\log\left(\frac{1+h}{h}\right) = \log(1+h)^{\frac{1}{h}}$ ok. So, check this. This essentially follows from the basic property of the logarithm and the way we have defined, the way we have defined exponentiating with respect to a real number, ok. Now, taking logarithms on both sides, immediately gives $\lim_{h \to 0} (1+h)^{\frac{1}{h}} = e$

So, this is yet another way of seeing this famous limit, this famous limit, this is a famous limit, ok. So, these are some basic properties of the logarithm functions; there are some more in the exercises. This is a course on Real Analysis, and you have just watched the module on the Logarithm.