Real Analysis - I
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Lecture – 31.2
The Inverse function Theorem

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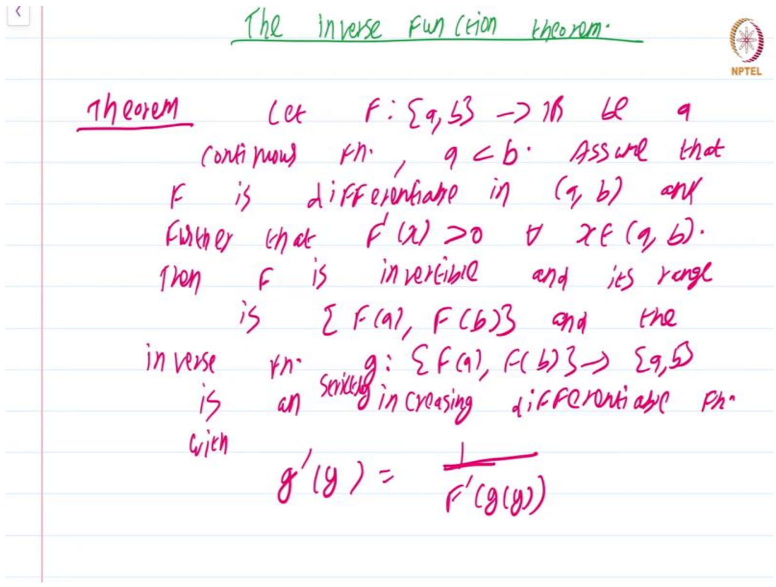

The Inverse function Theorem in several variables is one of the most used results from Real Analysis to branches like differential geometry. In this course, we are focusing on one variable. So, let me just state and prove the theorem in the one variable case. first, let us begin with a somewhat simplistic statement which we will in a moment generalize to the more general version.

Theorem, let $f: [a, b] \longrightarrow R$ be a continuous function, of course, $a < b$. Assume that f is differentiable in open interval (a, b) and further that $f'(x) > 0$ for all $x \in (a, b)$. Then f is invertible and its range is $[f(a), f(b)]$ and the inverse function $g : [f(a), f(b)] \longrightarrow [a, b]$ is an increasing differentiable function strictly increasing I might add, strictly increasing differentiable function with the derivative at a particular point $y = \frac{1}{f'(g(y))}$.

So, let me read out this statement. You have a continuous function from closed interval $[a, b]$ to R. We are assuming that f is differentiable in the open interval (a, b) and further that the derivative is greater than 0 at all points.

Now, we already know that under these hypotheses, the function f will be strictly increasing. Therefore, we have already shown that f is invertible in an earlier module and not only that we shown that f is invertible, but we have also shown that the inverse is a continuous function.

So, this inverse being a strictly increasing function is rather obvious. So, we have this inverse $g : [f(a), f(b)] \longrightarrow [a, b]$. The key assertion is that this inverse is differentiable and you have a formula for the derivative of the inverse.

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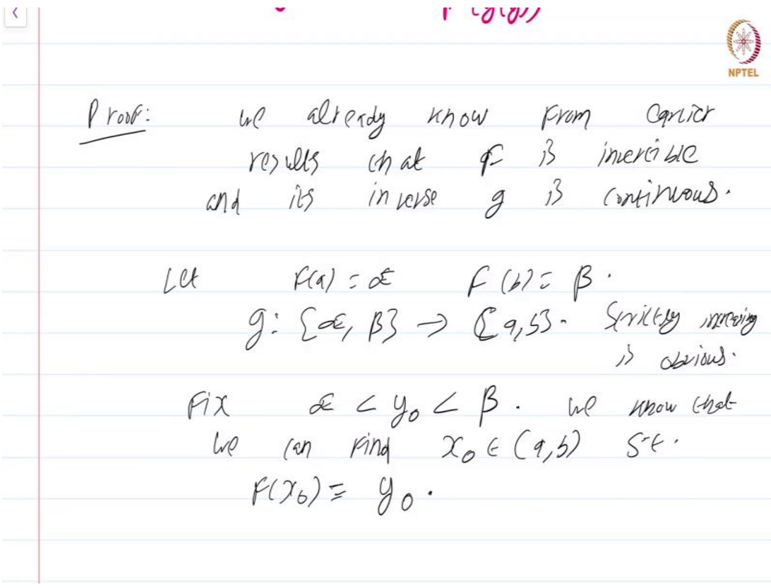

Now, the statement is long, but the proof is easy. Proof, we already know please check the notes for precise references to the earlier results. We already know from earlier results that that f is invertible and its inverse g is continuous. This much we already know. Well, what we do is the following.

Let $f(a) = \alpha$ and $f(b) = \beta$. We have this function $g : [\alpha, \beta] \longrightarrow [a, b]$. The fact that this is strictly increasing is obvious. Now, we have to show that g is differentiable on so, let me be ultra-precise differentiable function on open $\big(f(a), f(b)\big)$. Now, we have to show that g is differentiable so, what you do is fix $\alpha < y_0 < \beta$, we have to show differentiability at this point ok.

Now, because this function is going this function f is actually a bijective function, it is certainly injective and subjective we know that; we know that we can find; we can find $x_0 \in (a, b)$ such that $f(x_0) = y_0$. In fact, this point is unique because the function f will be strictly increasing because the derivative is greater than 0 everywhere. Again check precise references in the notes.

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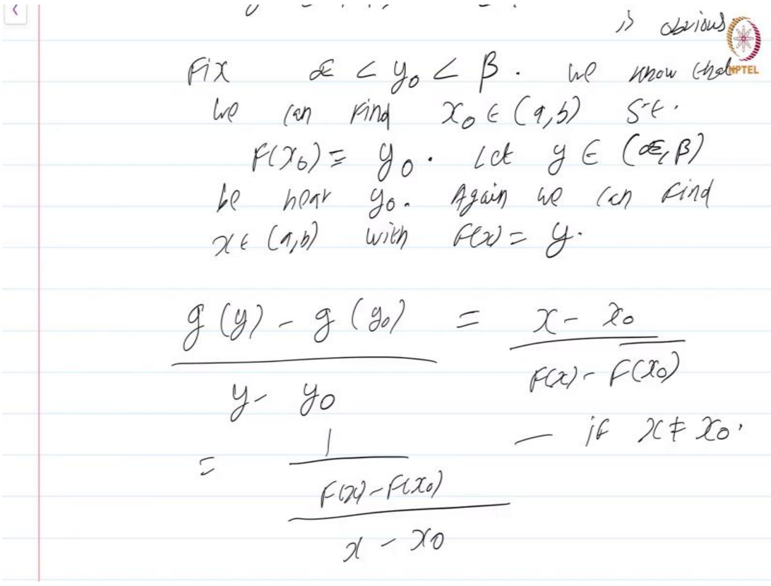

So, we have $f(x_0) = y_0$. Let $y \in (\alpha, \beta)$ be near y_0. Again, we can find; we can find $x \in (a, b)$ with $f(x) = y$. Now, all this is a setup to take the newton quotient $\frac{g(y)-g(y_0)}{y-y_0}$ and by the way things have been set up this is nothing but $\frac{x-x_0}{f(x)-f(x_0)}$.

We have set things up, so that this happens and this is nothing but $\frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}$. Of course, this is valid if $x \neq x_0$; all this is valid if $x \neq x_0$ ok.

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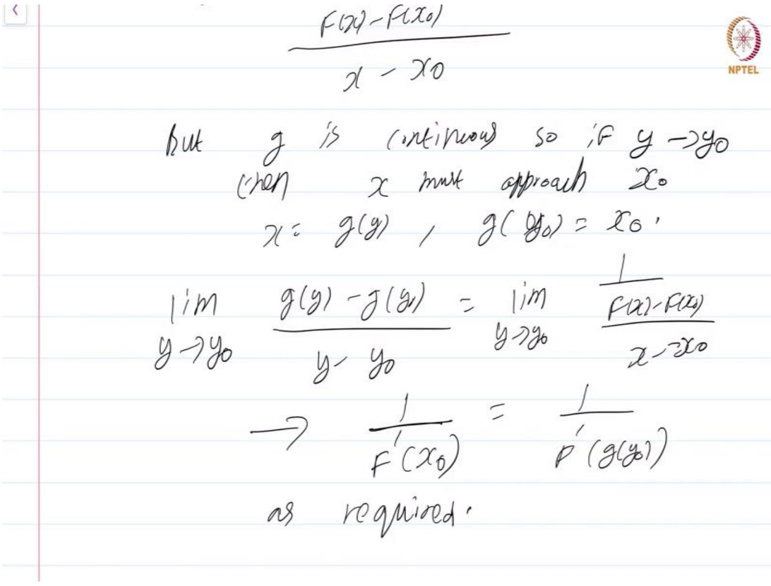

But g is continuous which was the how we began the proof g is continuous so, if y approaches y_0 , then this point x must approach x_0 simply because $x = g(y)$ and of course, $g(y_0) = x_0$.

So, $\lim_{y \to y_0} \frac{g(y)-g(y_0)}{y-y_0} = \lim_{x \to x_0} \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}$. from the remark, I just made that as y goes to y_0 x goes to x_0 so, this quantity inside just approaches $\frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}$, as required.

So, the proof is fairly straight-forward. You just compute the Newton coefficient. The only key part is that there is a function g that is inverse that just follows from the fact that f is strictly increasing which follows from the fact that the derivative is greater than 0.

So, this is typical of the proofs in mathematics. What we have done is over the course, we have proved several results essentially we are just combining all of them and this proof is immediate once you combine all of them ok.

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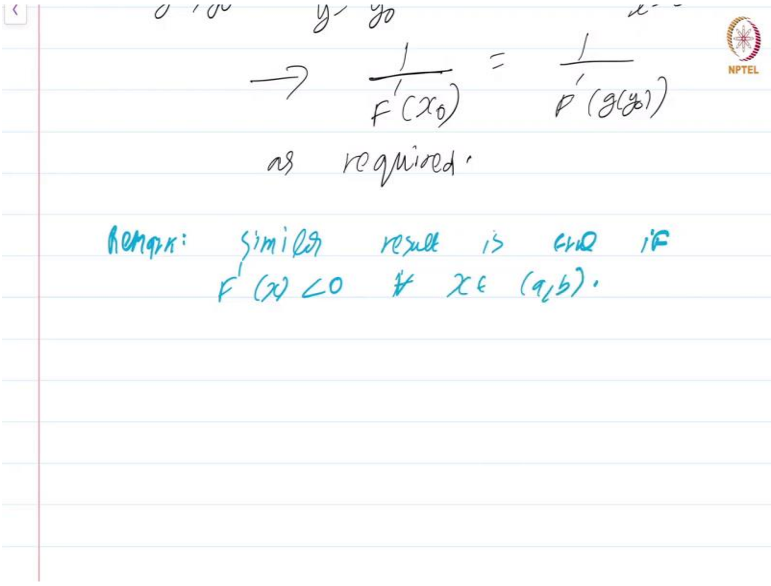

So, immediate remark similar result is true $f'(x) < 0$ for all $x \in (a, b)$. So, that means, f will be a strictly decreasing function then I had left an exercise for you earlier to show that in this case, actually the inverse of f actually exists and the inverse of f is going to be continuous so on and so forth, it is exactly the same proof will go through ok.

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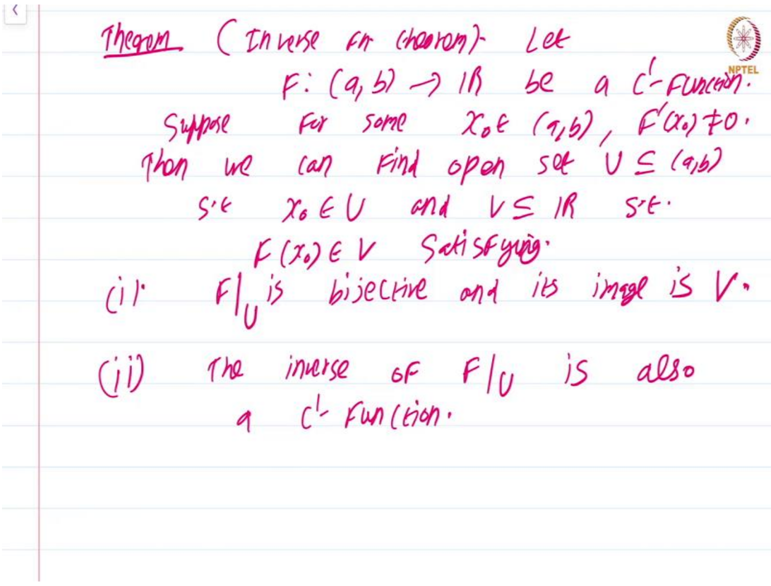

Now, I am going to state the general inverse function theorem; the general inverse function theorem and I am not going to prove it because it should be rather obvious from the theorem that we have already shown. This is the inverse function theorem. Let $f: (a, b) \longrightarrow R$

be C^1-function. Recall this means that the derivative of f exists and the derivative is in addition continuous.

Suppose for some $x_0 \in (a, b)$; $f'(x_0) \neq 0$; Then, we can find; we can find open sets $U \subset (a, b)$ such that $x_0 \in U$ and $V \subset R$ such that $f(x_0) \in V$ satisfying

(1). f is bijective or rather $f|_U$ is bijective and its image is V.

(2) the inverse of $f|_U$ is also a C^1-function.

Now, I am not going to prove this intentionally. The crux of this proof is already contained in the previous theorem. I want you to sit down and think about this and prove this theorem; it will give you a sense of satisfaction simply because this theorem in its several variable incarnations is one of the most used results in differential geometry. So, this concludes this module.

This is a course on real analysis, and you have just watched the module on the inverse function theorem.