Real Analysis - I

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Lecture – 31.1
The Exponential Function

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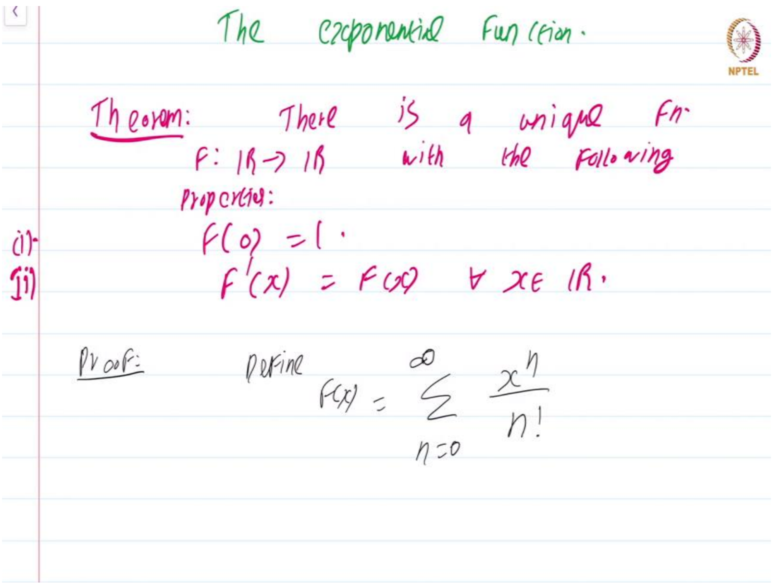

So far all our results have been general and abstract. We have not concentrated much on concrete functions except for a few modules that dealt with certain properties of polynomials and briefly with trigonometric and exponential functions.

In this last lap of this course, I am going to fix this lacuna. Let me first introduce the exponential function which is probably one of the most important functions from all of mathematics. Without further ado let me just state a theorem.

Theorem: There is a unique function $f: R \longrightarrow R$ with the following properties; with the following properties. (1). $f(0) = 1$. (2). $f'(x) = f(x)$ for all $x \in R$.

So, there is a unique function which takes the value 1 at 0 and whose derivative is itself.

Proof, now that we are masters of power series this is a fairly easy thing to do. Define $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

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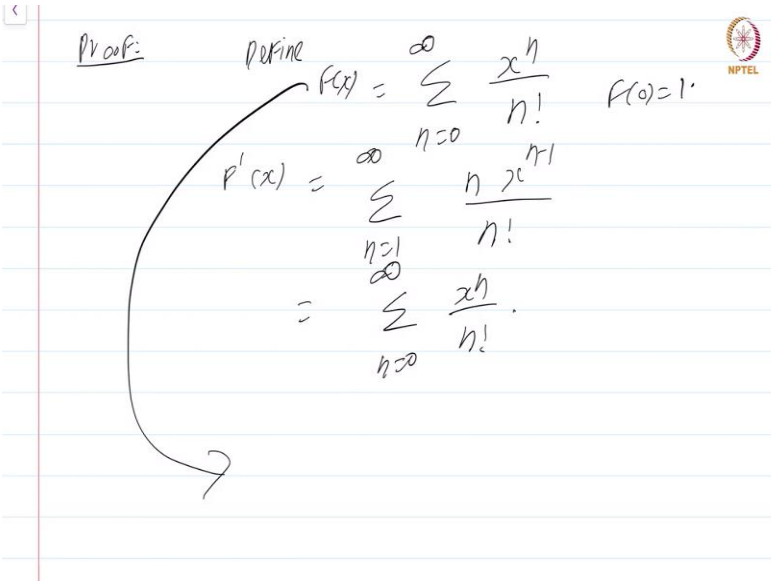

Now, the question is if this series converges then what is its derivative? Let us first assume that the series converges, let us compute its derivative, well, $f'(x)$. We know that if a power series converges, then in the interval of convergence, you can do term by term differentiation.

So, this will be $\sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!}$ which you can just see is just $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and it is also clear that f(0) =1, that is also trivial to see. So, all we have to show is that this function f converges at every point of the real numbers.

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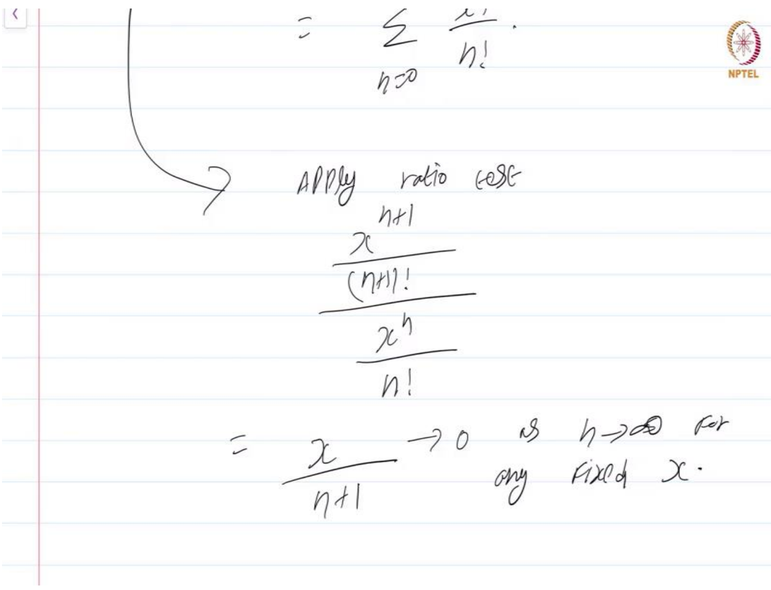

Now, there are several proofs of why this series converges, I suggest that you spend a nice monsoon afternoon contemplating various proofs. Let me give the simplest, not the one that uses the least number of tools, but the simplest according to me, apply ratio test.

Well, what do you get when you apply ratio test? You get $\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}$, well what happens to this, you just get $n!$ and $(n+1)!$, the n! part will get cancel you will get $\frac{x}{n+1}$ which of course, converges to 0 as n approaches infinity for any fixed x.

Therefore, for each x, the ratio test tells you that it is convergent. Therefore, the power series converges at every point of the real numbers. In other words, the radius of convergence of the power series is infinity.

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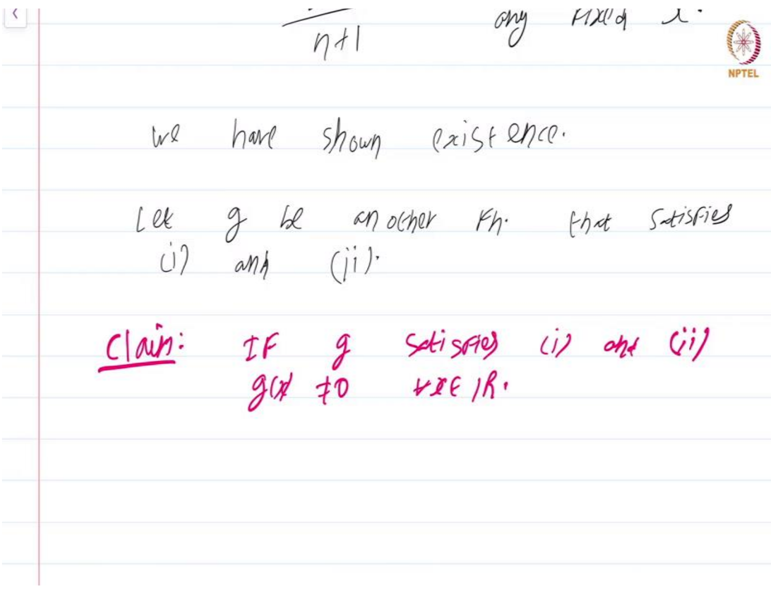

So, we have shown existence. Well, what about uniqueness of this power series? Well, what you do is let g be another function that satisfies one and two. We have to show that $f = g$. How are we going to do that first we will show that any function that satisfies one and two is never 0. Claim if g satisfies one and two, $g(x) \neq 0$ for all $x \in R$.

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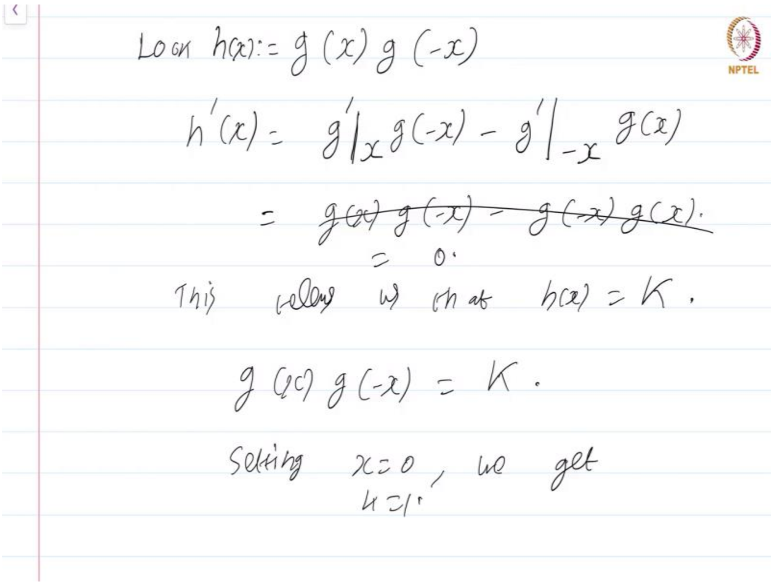

Well, why is this the case? Well, because look at $g(x)g(-x)$, look at this new function call it $h(x)$, differentiate this h. Well, by product rule what do you get? You get $g'(x)g(-x) - g'(-x)g(x)$. I hope, I have done the differentiation correctly.

Well, g', one moment, this is just if this has to be 100 percent precise, let me just clarify because there might be a bit of confusion if I write it like this, this is derivative of g evaluated at the point x so, I am writing bar and x at bottom to sort of say that I am first taking the derivative, then evaluating at the point x times $g(-x)$. I am just applying the product rule, then I have to differentiate $g(x)$ which by chain rule is just $-g'$ so, that is why there is a minus sign. So, it is $-g'$, but this time evaluated at $-x$; $-g'(x)|_{-x}g(x)$.

But we know that $g' = g$, right. So, this is just $g|_x g(-x) - g|_{-x} g(x)$ ok. So, the reason why I introduced this notation of evaluating at the point x is to make sure that when I write $g'(-x)$, you do not think it is I have to differentiate the function $g'(-x)$ which is $-g'(-x)$ which is which will sort of lead to confusion. So, that is why I am introducing this notation. Now this everything just cancels and you get 0 ok.

So, what does this tell us? Well, this tells us that $h(x)$ is a constant is some constant let us say k which means $g(x)g(-x) = k$ and wait a second we can even determine what this constant is determine what this constant is setting k = 0 sorry setting $x = 0$, we get $k = 1$.

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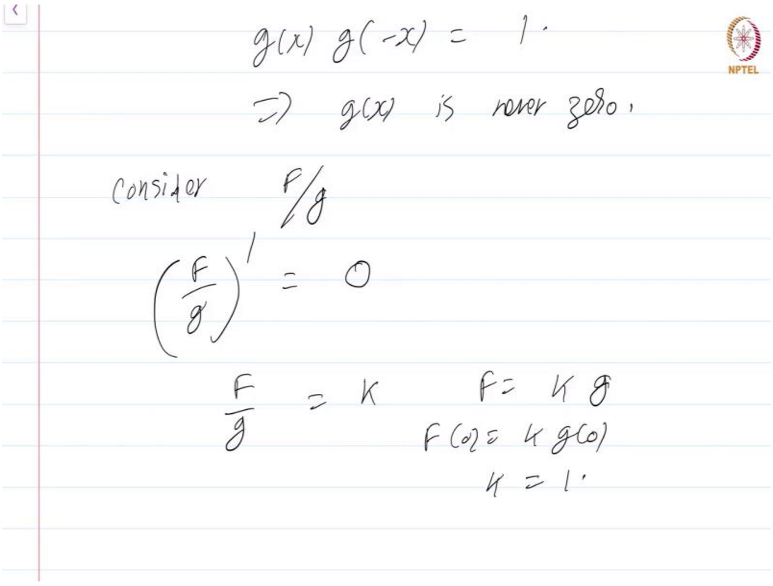

So, $g(x)g(-x) = 1$ which means $g(x)$ is never 0. So, this proves the claim that any function that satisfies properties one and two is never 0. Now, what we do is consider $\frac{f}{g}$, consider the function $\frac{f}{g}$. f and g are both functions that satisfy the properties one and two. Now, differentiate $\frac{f}{g}$, differentiate this.

Now, I am not going to bother doing the derivative. Just by the fact that $f' = f$ and $g' = g$ you will immediately conclude that the derivative is 0 because the numerator will have $fg' - f'g$. So, this will be 0. So, what this shows is that $\frac{f}{g}$ is some constant k. So, $f = kg$. Now, that means, $f(0) = kg(0)$ or in other words, $k = 1$.

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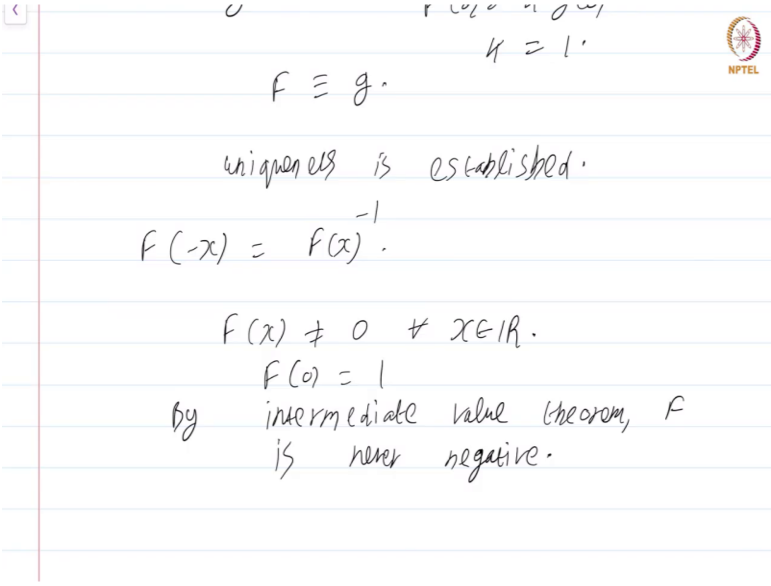

So, we have shown that f is identically equal to g ok. So, uniqueness is established; uniqueness is established. We have also got a useful fact about this function f which came out in the proof, the fact that $f(-x)$ is just $f(x)^{-1}$. Please keep this in mind this will be very very useful ok.

Now, we already know that $f(x) \neq 0$ for all x that is also something that came out in this proof and we also know that $f(0) = 1$. Therefore, by intermediate value theorem because f is of course, a continuous function it is in fact, given by a power series so, it is a smooth function by intermediate value theorem; by intermediate value theorem f is never negative. So, the function f whose existence and uniqueness we have established in this theorem also is always a positive function.

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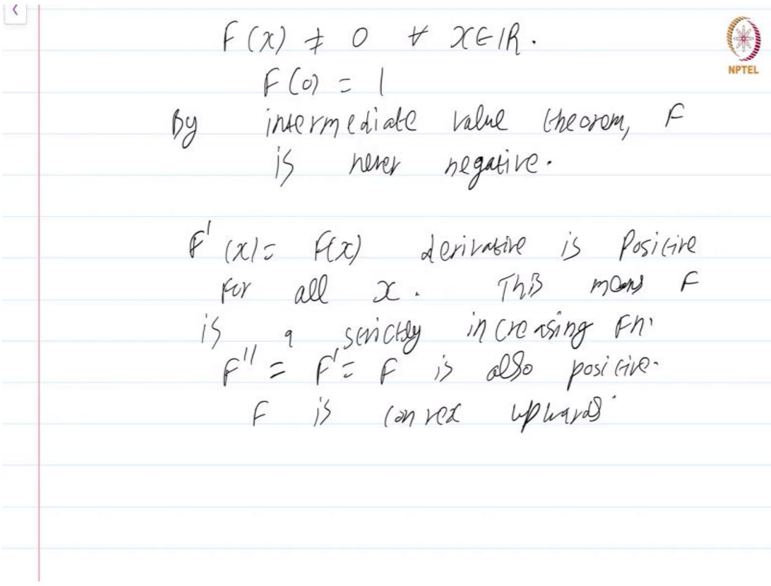

Now, $f'(x) = f(x)$, right which means derivative is positive for all x right, the derivative is a positive function. This means f is a strictly increasing function which we have seen in the various theorems that we have established in the part on calculus on differentiation the part on differentiation.

So, f is a strictly increasing function. Not only that $f'' = f' = f$ is also positive; is also positive which just goes to show that f is convex upwards.

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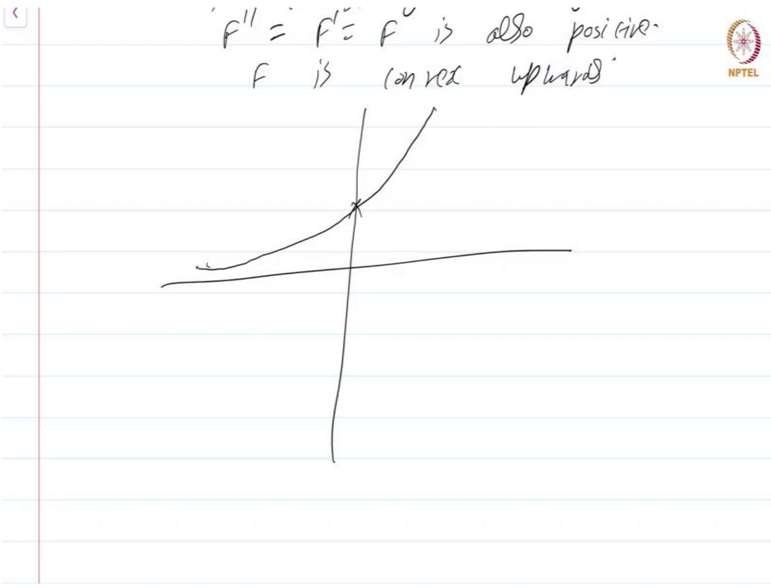

Putting all of this together, we get the familiar graph of the exponential function at 0 it is 1 and the function is sort of going to look like this. So, I have drawn it badly it should actually be more steep; should actually be more steep.

So, let me try to rectify, I will not say I will rectify it for sure something like this should be steeper, this is the roughly the graph of the exponential function as we can see from all the facts that we have the fact that it is strictly increasing, it is convex upwards at 0 it is 1 so, on and so forth ok.

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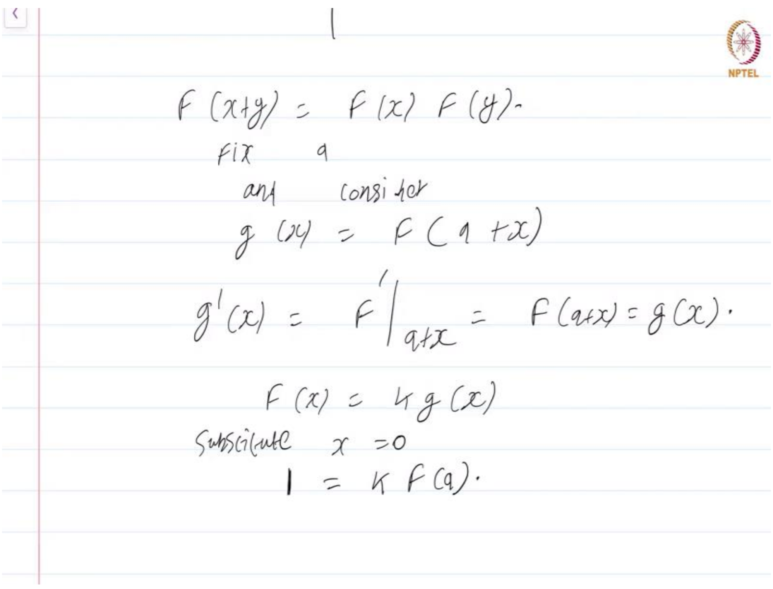

Now, another property of the exponential function that you are familiar with is $f(x + y) = f(x)f(y)$. Now, you will notice that I am going to prove everything from the two properties in the previous theorem that $f(0) = 1$ and $f' = f$. So, what I will do is fix a and consider $g(x) = f(a + x)$

Now, differentiate. $g'(x) = f(a + x)$; $f(a + x)$ times what is inside sorry it is $f'(a + x)$ times the derivative of what is inside by chain rule, but the derivative of what is inside is just 1. So, $g'(x) = f(a + x)$. Again, for clarity let me just write; let me just write $f'|_{a+x}$; which is equal to $g(x)$.

Now, if you go through the argument that we gave to show that if there is another function g that satisfies both one and two, then we must have that $f = kg$ right and then, we used property 1 to show that k is 1. So, here we have a function g that is just satisfies property 2.

The fact that $g' = g$ with just that property alone we will be able to show that $f(x) = kg(x)$, that we will be able to show. We will not be able to show that this k is equal to 1 because we do not know that $g(0) = 1$, but this is a start.

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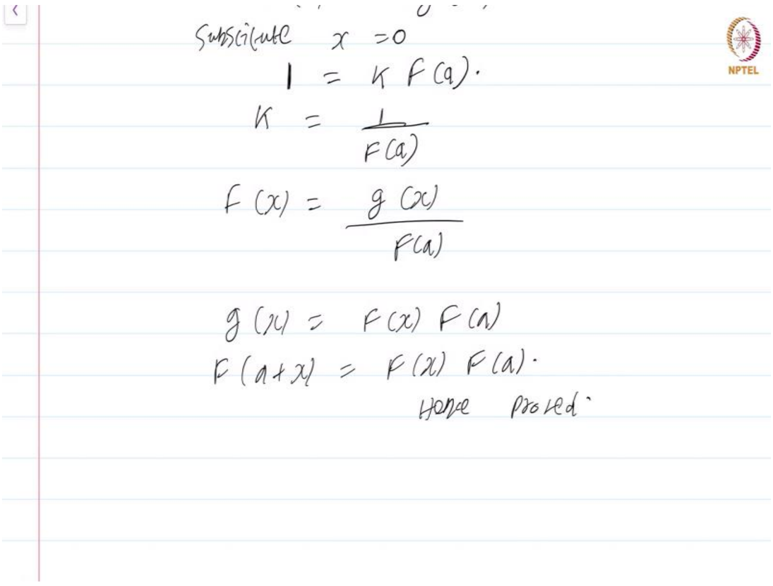

Now, substitute $x = 0$ and let us see what happens. In the previous proof, k dropped out to be 1 substitute $x = 0$. So, what you get is $1 = kf(a)$right because we have substituted $x = 0$ and $g(x) = f(a + x)$ so sorry this should be; this should be $f(a)$ not g(a) this should be $f(a)$ fine.

Which just goes to show that k; $k = \frac{1}{f(a)}$, that means, what we get is $f(x) = \frac{g(x)}{f(a)}$ or in other words $g(x) = f(a)f(x)$ which just goes to say that $f(a + x) = f(x)f(a)$, hence proved.

So, we ultimately wanted to show that $f(x + y) = f(x)f(y)$, we have just fixed $y = a$ and proceeded with this argument. So, ultimately, we have got the product property of exponential functions.

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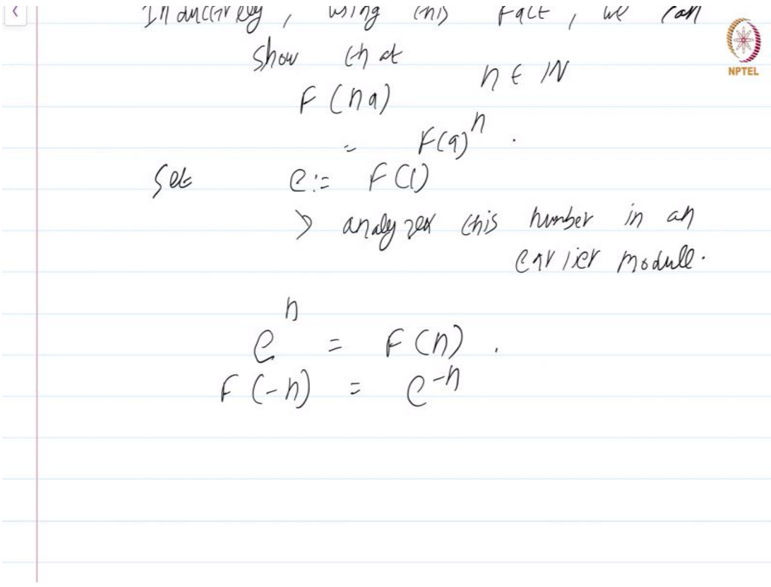

Now inductively using this fact we can show; $f(na) = f(a)^n$ $n \in N$ ok. This is certainly true when $n = 1$. When $n = 2$, we just say $f(a + a) = f(a)^2$, that is what the previous result is saying inductively it is easy to show this.

Now, set $e = f(1)$, define $e = f(1)$. We have already analyzed; analyzed this number in a module, in an earlier module. In particular, we have shown that this e is actually an irrational number, we have actually shown that this e is an irrational number.

So, what our theory so far says is that $e^n = f(n)$; excellent. We also know that $f(-n) = e^{-n}$ this also follows from the fact that $f(-x) = f(x)^{-1}$ because of that we get $f(-n) = e^{-n}$.

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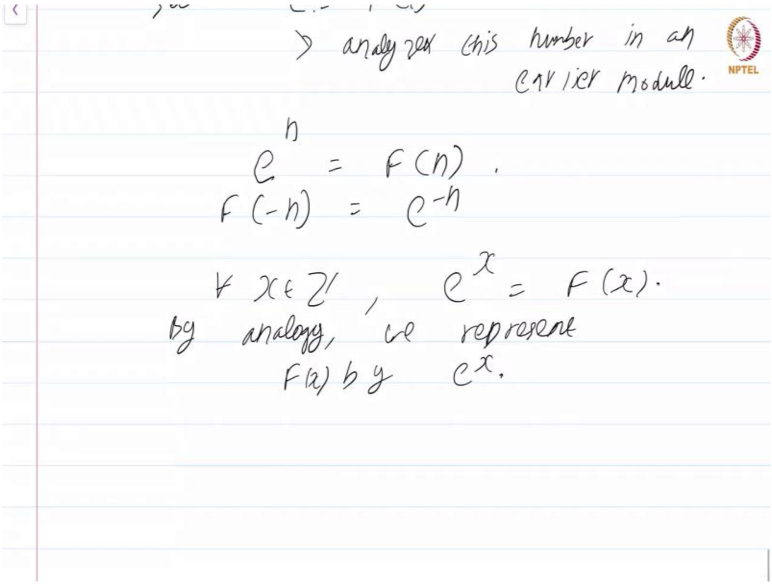

So, what we have got is at least for certain quantities; at least for certain quantities just a moment this is not there is no a in these equations, it is just sorry about this, $e^n = f(n)$ and $f(-n) = e^{-n}$. So, at least for all integers for all $x \in Z$, we have this $e^x = f(x)$. So, by analogy we represent $f(x)$ by e^x.

Note very carefully what is going on. I am not saying that the function f whose existence and uniqueness we have proved is actually exponential with base e^x that is not what I am saying. We have not yet defined what power x means when x is just any real number. We know how to define exponential taking power to a natural number, we also know how to do it for a rational number, but we do not yet know how to do it for a real number.

We have just informally defined it using logarithms as and when needed in earlier modules, but the purpose of this particular set of modules is to make all that rigorous ok. So, by analogy we are representing $f(x)$ by e^x that is all; that is all we are doing.

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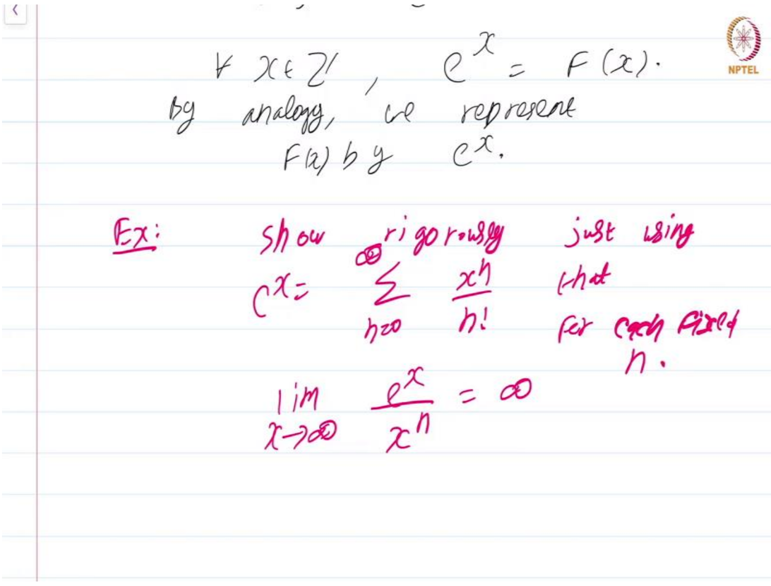

Now, I am going to leave you with an exercise; I am going to leave you with an exercise. Show rigorously just using $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ that $\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$ so, for each fixed n. We have already established this, but I want you to show this just by using the power series expansion ok; just by using the power series expansion.

So, these are all the properties of exponential that will be needed in this I mean that is not just needed that is usually needed in practice the exercises contain more properties of the exponential function that you can prove either directly from the power series expansion or better just by using properties 1 and 2 in the big theorem we proved in the beginning of this module.

This is a course on real analysis, and you have just watched the module on the exponential function.