Real Analysis - I

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Lecture – 29.2
Definition of Uniform Convergence

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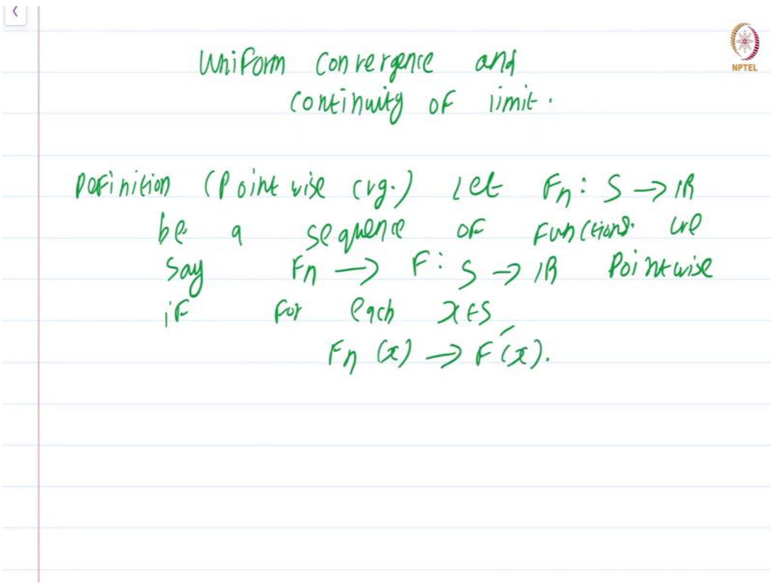

Let us begin with the definition. Definition, this is of point wise convergence. Let $f_n: S \longrightarrow R$R be a sequence of functions of functions. We say we say f_n converges to the function $f: S \longrightarrow R$ point wise point wise if for each $x \in S$, $f_n(x)$ converges to $f(x)$ ok. In the last module, we have already seen an example of such convergence, we saw that this function x^n converges to the function that is 0 if $x \in [0,1)$ and 1 if $x = 1$, we already saw this.

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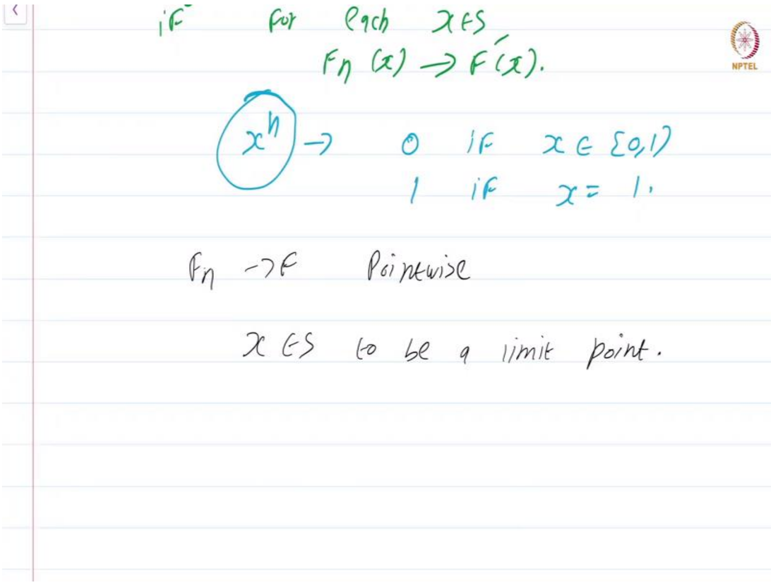

So, we know that this point wise convergence of a sequence of functions is not a guarantee that the limit function will also be continuous, even if you start with the nicest of nice functions x^n.

This is this need not happen. So, what we are going to now do is to reverse engineer and try to find out some appropriate condition that will guarantee that the limit function is continuous. To do that, let us try to see what goes wrong if we naively try to prove that the limit function is continuous.

So, what we are going to do is we are going to consider f_n converging to f point wise; we have point wise convergence f_n converging to f. We are going to take $x \in S$ to be a limit point and try to show that the limit function f is actually going to be continuous at x and fail miserably. At the isolated points of the set as f is anyway continuous, so I do not care about those points.

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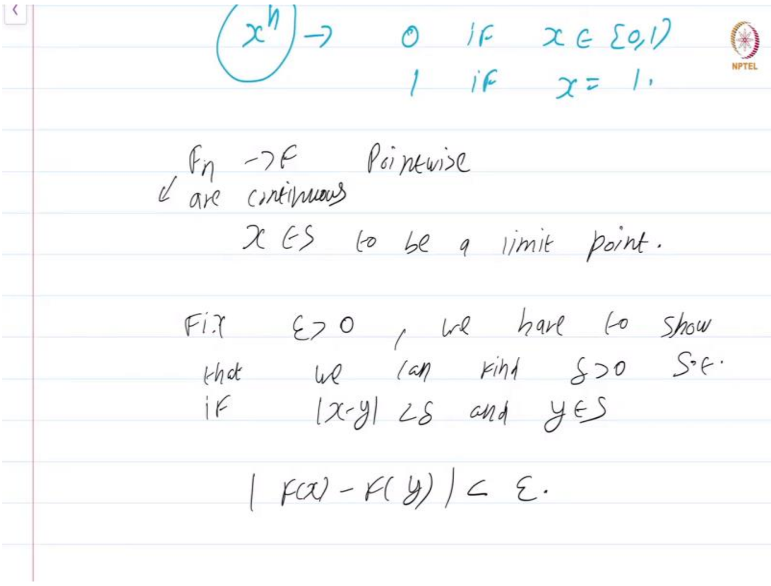

All the issues will come at the limit points ok. So, what we are going to do is the following. What we have to do is fix $\varepsilon > 0$, we have to show that we can find $\delta > 0$ such that if $|x - y| < \delta$ and $y \in S$, $|f(x) - f(y)| < \varepsilon$. This is what we have to do ok.

Now, of course, I must mention f_n's are continuous. These functions f_n are continuous otherwise this whole enterprise is doomed to fail right from the beginning ok. So, now, that we want to show $|f(x) - f(y)| < \varepsilon$ and the only data, we are given is about the continuity of the functions f_n and the fact that f_n converges to f the natural thing to do is to play the oldest trick in the book and introduce f_n's by adding and subtracting.

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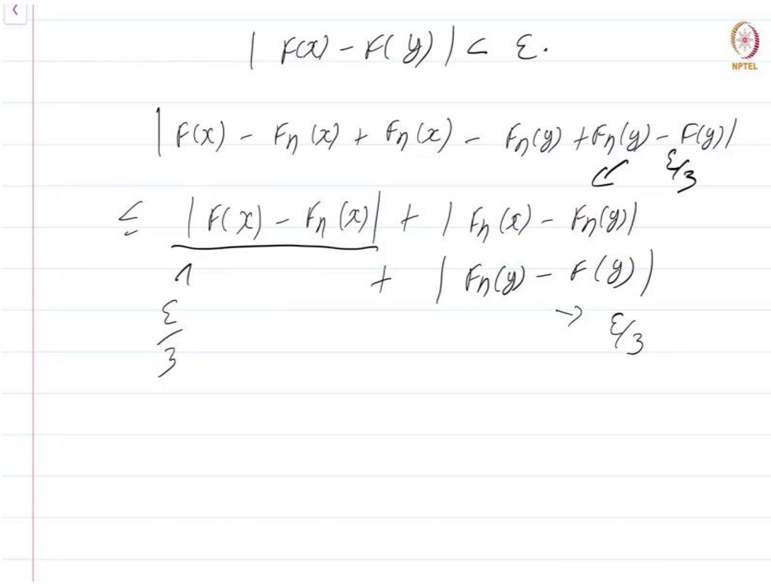

So, what we can do is we can add and subtract $f_n(x)$ ok. Then, we can add and subtract $f_n(y)$ and then, let us expand this out by triangle inequality, we will get less than or equal $\left|f(x) - f_n(x)\right|$+$|f_n(x)$–$f_n(y)| + |f_n(y) - f_{(y)}\right|$ ok. Now, this splitting up term to get three terms in this manner repeatedly arises in analysis.

So, be very careful and try to understand what exactly is going on ok. Now, if you look at these terms carefully and ponder for a few minutes, you might be deluded into thinking that actually this professor is saying nonsense; the limit function f is certainly continuous. Why? Because think about this. $|f(x) - f_n(x)|$ can be made less than $\frac{\varepsilon}{3}$; simply because $f_n(x)$ is converging to $f(x)$.

The middle term $|f_n(x)$–$f_n(y)|$ $c$an be made less than $\frac{\varepsilon}{3}$ by choosing δ appropriately right. Because f_n*'s* are continuous, you can choose a δ so that $|f_n(x)$–$f_n(y)| < \frac{\varepsilon}{3}$.

And of course, the third term can be made less than $\frac{\varepsilon}{3}$ for exactly the same reason $f_n(y)$ converges to $f(y)$. So, you would think that it is certainly possible to make $|f(x) - f(y)| <$ ε by choosing δ appropriately, but not so quick.

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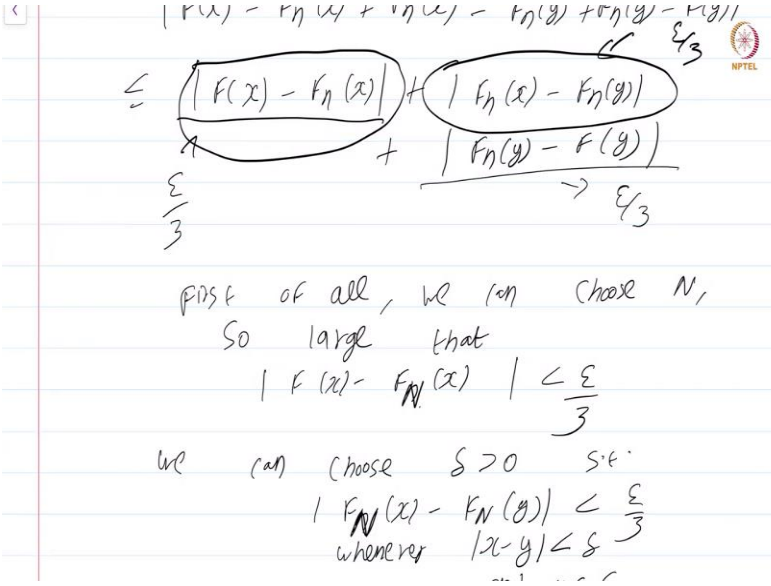

Let us see what happens. First of all, we can choose capital N so large that $|f(x) - f_N(x)|$, of course this should be capital N is less than $\frac{\varepsilon}{3}$. We can do this simply because the sequence $f_n(x)$ converges to the point $f(x)$. We can choose capital N so large that $|f(x) - f_N(x)| < \frac{\varepsilon}{3}$. So, the first term, we can handle.

Now, what about the middle term? Well, we can choose $\delta > 0$; we can choose delta greater than 0 such that $|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$ whenever $|x - y| < \delta$ and $y \in S$. We can do this.

So, what we have done is we have controlled this first term by controlling capital N. I mean small n and setting it to be a large enough number capital N, then we are controlling the second by the continuity of f_N. Now, what about the third term? The third term is now we have to control $f_N(y) - f(y)$.

Now, here is where the issue lies; y could be any point such that $|x - y| < \delta$ and $y \in S$ and the issue arises because just by ensuring that $|f(x) - f_N(x)| < \frac{\varepsilon}{3}$ does not give us any control on $|f_N(y) - f(y)|$. We might have to bump up this capital N. We have no control over y.

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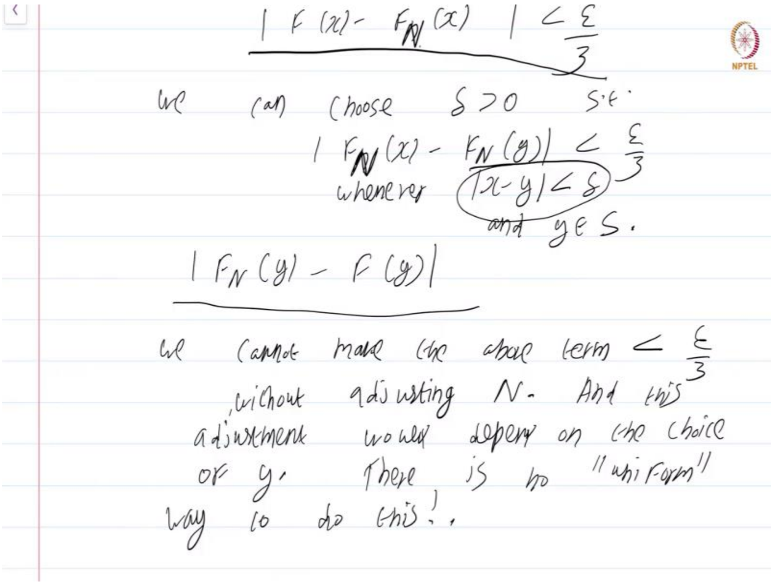

So, depending on the choice of y which is now contained in this which is δ close to x, we have to choose capital N possibly higher depending on the choice of y. So, we cannot make the above term less than capital N; sorry $\frac{\varepsilon}{3}$ without adjusting without adjusting capital N.

And this adjustment would depend on the choice of y. In other words, there is no uniform way to do this ok. So, this discussion motivates the following definition. The definition is just modeled. So, as to make the above flawed proof not flawed.

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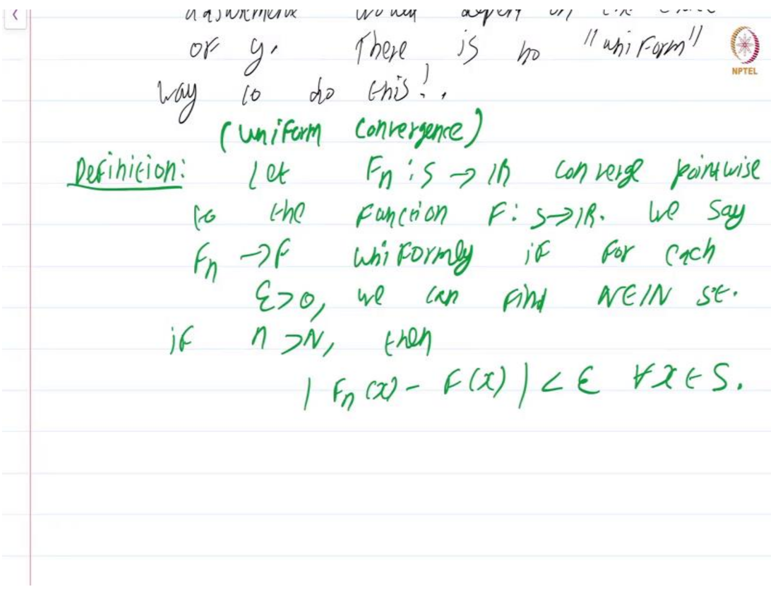

So, let $f_n: S \longrightarrow R$ converge point wise, point wise to the function $f: S \longrightarrow R$. We say f_n converges to f uniformly. So, uniform convergence is stronger than point wise convergence, if for each ε > 0, we can find capital $N \in N$ such that if $n > N$ is greater than capital N, then $|f(x) - f_N(x)| < $ ε for all $x \in S$.

The previous definition of point wise convergence merely said that $f_n(x)$ converges to $f(x)$. Therefore, the choice of N would crucially depend on the choice of x; whereas, in this definition of uniform convergence, so this is the definition of uniform convergence; the choice of N does not depend on the choice of the point x. You can find a uniform N that works simultaneously for all the points $x \in S$ ok.

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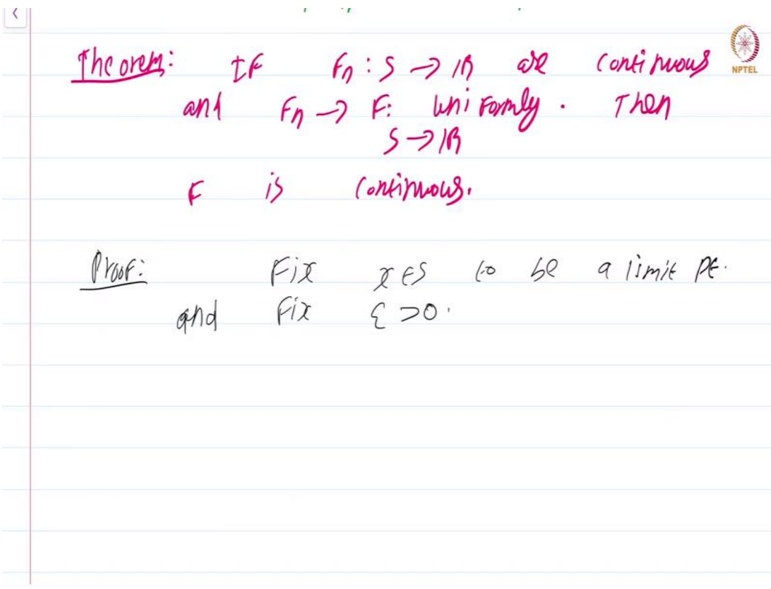

So, let us immediately prove a theorem and this theorems proof should be fairly clear. If $f_n: S \longrightarrow R$ are continuous and $f_n(x)$ converges to $f(x)$ uniformly; of course, $f: S \longrightarrow R$. Then, f is continuous ok. Let us see the proof; most of the work is done. Let us see the proof fix $x \in S$ to be a limit point and fix ε > 0.

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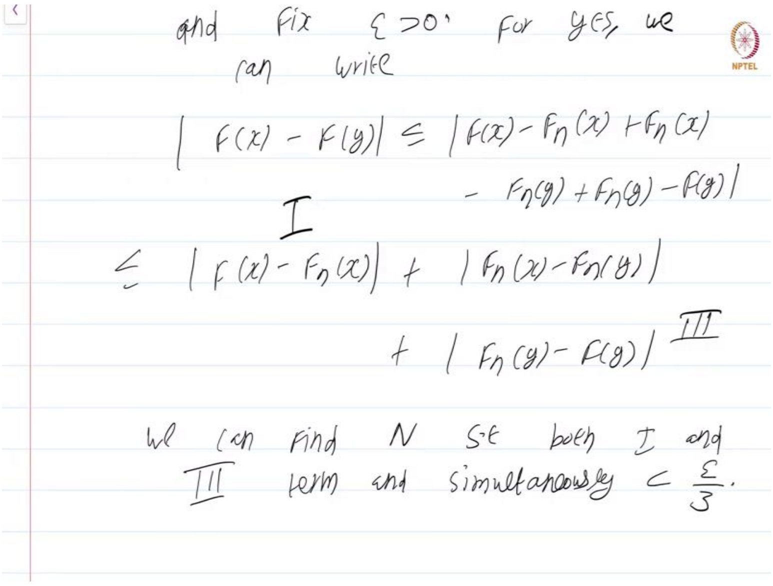

Again, we can write for $y \in S$, we can write $|f(x) – f(y)|$ less than or equal to as usual exactly as we done as we did in the preceding discussion, $|f(x) – f(y)| \leq |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)|$ and this just becomes$|f(x) - f_n(x)| + |f_n(x) - f_n(y) + |f_n(y) - f(y)|$.

Now, here is the crucial fact, we can find; we can find capital N such that both the first term first and third term are simultaneously less than $\frac{\varepsilon}{3}$ ok. It does not matter no matter what points x and y you choose in the set S, you can always find a capital N such that $|f(x) – f_n(x)| < \frac{\varepsilon}{3}$ $|f_n(y) – f(y)| < \frac{\varepsilon}{3}$.

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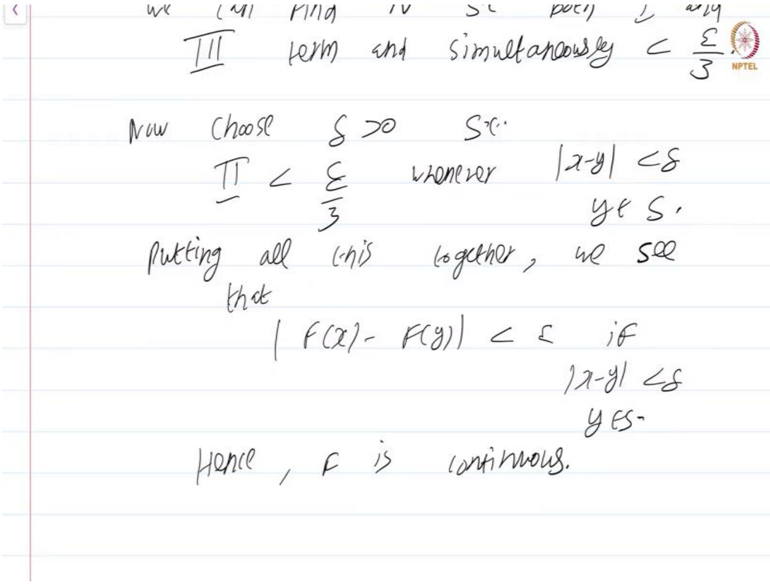

So, these first and third terms can be made less than $\frac{\varepsilon}{3}$ simultaneously; it does not matter where x and y are ok. Now, choose $\delta > 0$ such that the middle term too is less than $\frac{\varepsilon}{3}$ whenever $|x - y| < \delta$ and $y \in S$ ok.

So, putting all this together, putting all this together, we see that $|f(x) - f(y)| < \varepsilon$, if $|x - y| < \delta$ and $y \in S$. Hence, f is continuous ok. So, the definition of uniform convergence was built precisely to make this argument work.

Now, in the next few modules, we will explore certain properties of uniform convergence; how it behaves, how uniformly convergent sequence of functions behave under differentiation under integration and so on. Then, we will apply these results to power series. It will turn out that any power series that converges in a particular interval will do so uniformly with a slight twist, we will see more about this in the future modules.

This is a course on real analysis, and you have just watched the module on uniform convergence and continuity of limits.