Real Analysis - I

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Lecture – 29.1
Power Series

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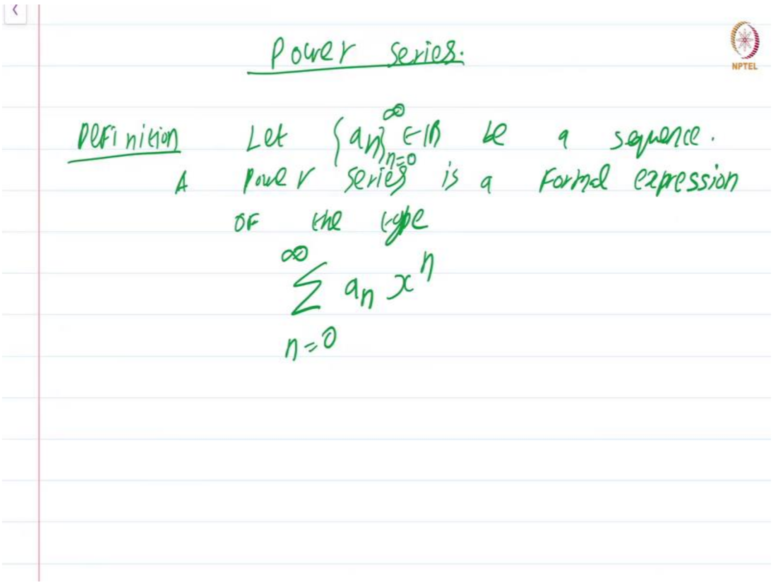

We are now in the final lap of this course, we are going to be studying the topic of uniform convergence (Refer Time: 0:22) Power Series. In fact, the topic of power series motivates why we need to study uniform convergence. So, far in this course, we have tackled many difficult theorems; but it is somewhat disturbing that the only real examples of smooth functions, differentiable functions that really we have understood to some degree are the polynomials.

I have invoked sine, cosine, exponential here and there in the passing as examples; but we have not really studied how they are defined and a deep analysis of their properties is not yet done. The correct way in my humble opinion of how to define these common elementary functions is via power series.

So, let us begin with the definition of a power series definition. Definition: let $(a_n) \in R$ be a sequence be a sequence; a power series is a formal expression of the type $\sum_{n=0}^{\infty} a_n\, x^n$. So, this sequence a_n is actually from $n = 0$ to ∞. So, this is a sequence that is starting at the term 0, ok.

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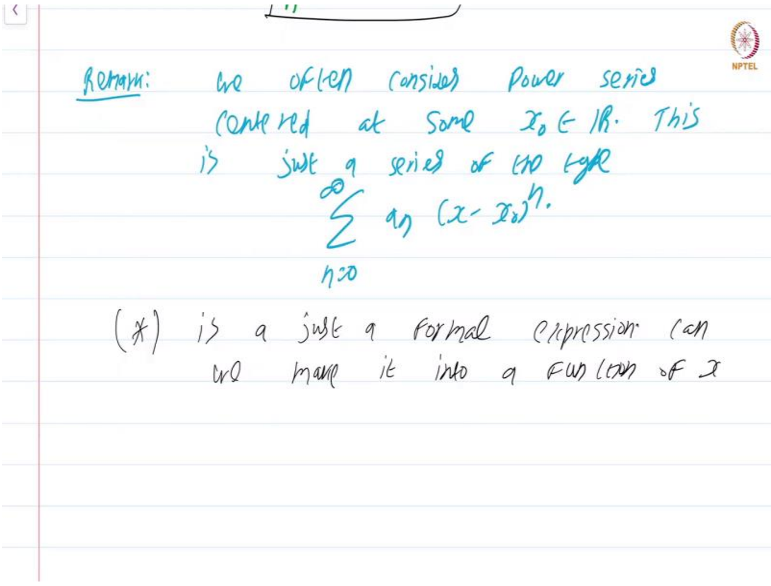

So, let me just make a remark, we often consider power series centered at some $x_0 \in R$. This is just a series of the type $\sum_{n=0}^{\infty} a_n (x - x_0)^n$.

However, the original definition of power series that we have given, the theory that we will develop to study such series is more than sufficient to deal with more these more general series centered at some point x_0. It is simpler to study just when the center is just the origin and the theory in the general case is straightforward to generalize, ok.

So, for simplicity, I will just concentrate on series of this type. Now, the first question you should be asking is, let us call this series star, is just a formal expression. Can we make it into a function of x?

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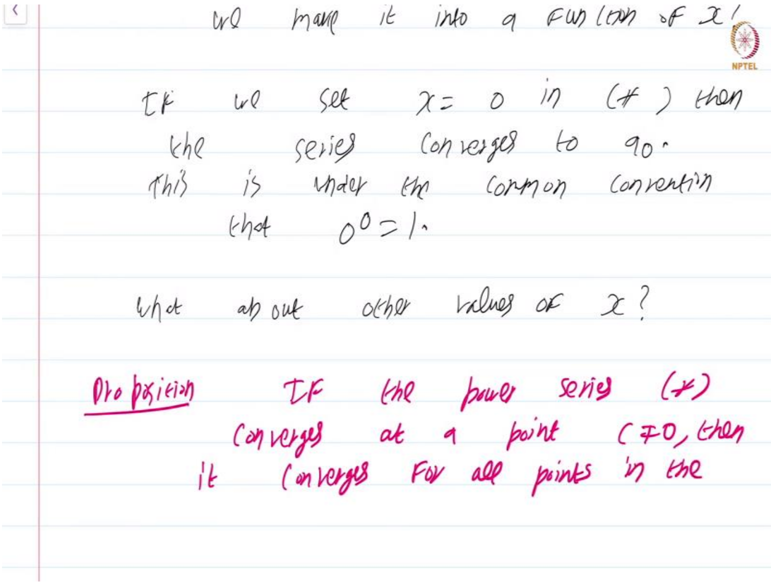

If we set $x = 0$ in star, then the series converges to a_0 . Now, of course if you look at the series carefully and if you have the type of listener who is very very eagle eyed; you will notice that when I say that this series converges to a_0, I am making an assumption about 0^0.

So, I must write this is under the common convention $0^0 = 1$; only under this convention does it happen that star converges to a_0, ok. So, in any case under this convention, we know that when you substitute $x = 0$; the series indeed converges and it converges to the coefficient a_0. What about other substitutions, what about other values of x?

Thankfully, power series behave very nicely with respect to the set of points where the series converges; that is dealt with in the following proposition. Proposition: If the power series power series star converges at a point $c \neq 0$; then it converges for all points all points in the set $\{x: |x| < c\}$, ok.

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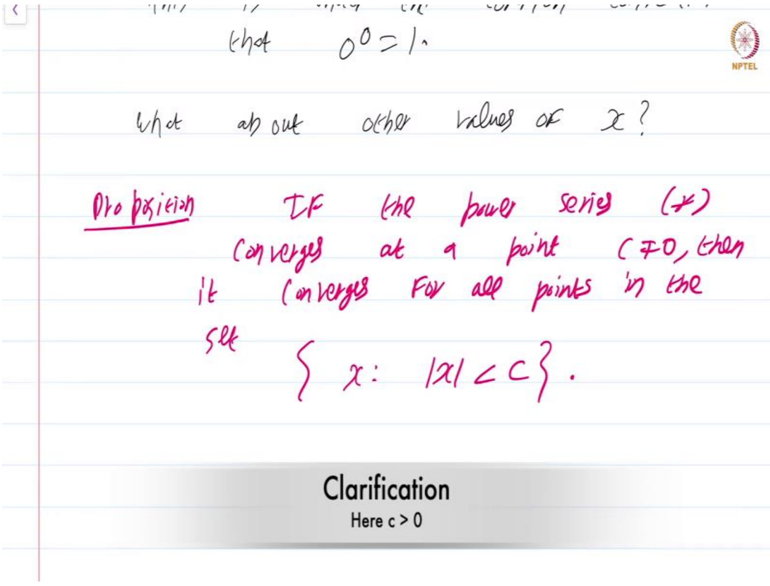

So, if it happens that at a particular substitution $x = c$, $c \neq 0$, the series does converge; then it converges for all the points x on the real line whose absolute value is actually less than c.

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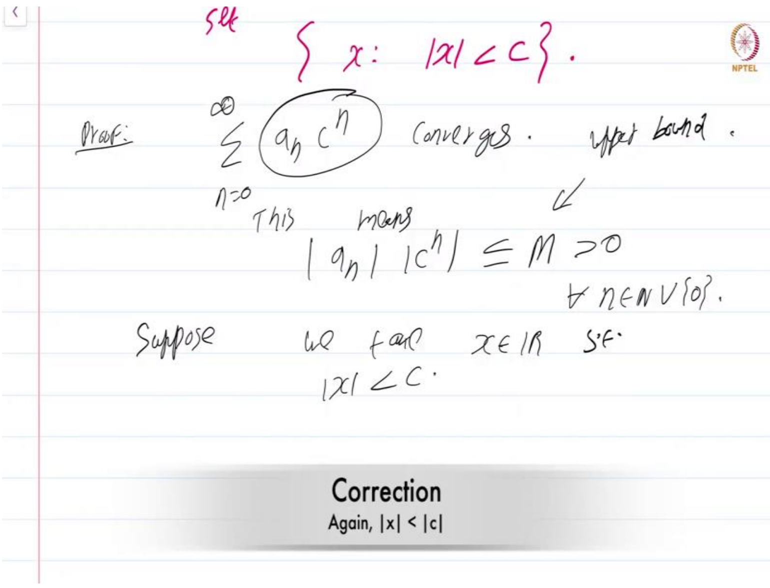

Let us see a proof of this, this is a nice application of the comparison test; one of the most powerful tests for studying convergence. So, let us see. So, we have that $\sum_{n=0}^{\infty} a_n c^n$ converges, this is the assumption, ok. Now, this means first of all that, $|a_n||c^n| \leq M > 0$, some upper bound.

Since the sequence converges to 0 and therefore, the sequence has to be bounded. When I say sequence, I mean the sequence $a_n c^n$, not just the sequence a n, ok.

So, the sequence must be bounded, therefore we can find an upper bound capital M, such that mod $|a_n||c^n| \leq M$, which is a quantity that I am taking to be greater than 0 and this is true for all $n \in N \cup \{0\}$, ok. How does this help us? Well, suppose we take $x \in R$, such that $|x| < c$, ok. Suppose we take a point c, where we take a point $x \in R$ whose absolute value is less than c.

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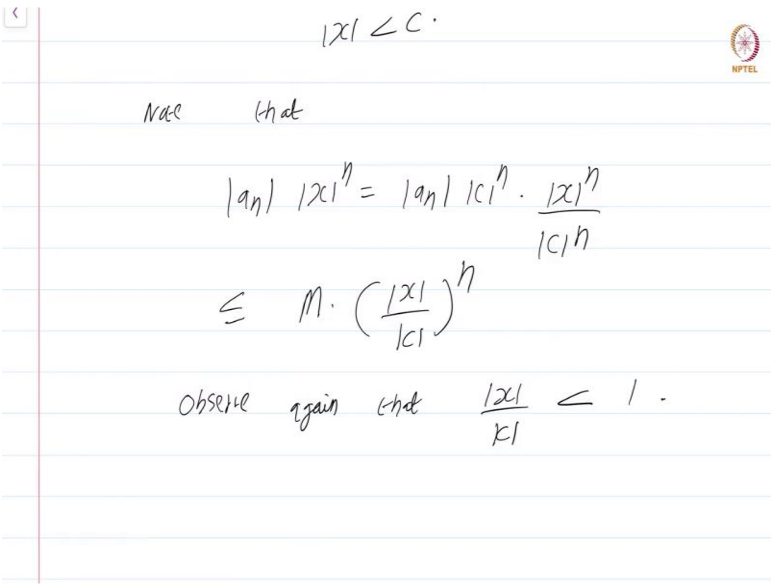

Now, note that, $|a_n||x|^n = |a_n||c|^n \frac{|x|^n}{|c|^n}$, ok. I have just added and has not added; multiplied and divided by $|c|^n$. And by our assumption that mod $|a_n||c|^n \leq M$; we get less than or equal to $M \left(\frac{|x|}{|c|}\right)^n$, ok. But observe again that, $\frac{|x|}{|c|} < 1$.

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$$|a_n|\,|x|^n = |a_n|\,|c|^n \cdot \frac{|x|^n}{|c|^n}$$

$$\le M \cdot \left(\frac{|x|}{|c|}\right)^n$$

Observe again that $\frac{|x|}{|c|} < 1$.

By comparison test the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely!.

So, by comparison test by comparison test, the series the series $\sum_{n=0}^{\infty} a_n\, x^n$ absolutely. So, we got a stronger conclusion, we got a stronger conclusion than claimed. So, we can fix this claim converges absolutely for all points in the set $\{x: |x| < c\}$.

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Proposition: If the power series (*) converges at a point $c \neq 0$, then it converges absolutely for all points in the set $\{x : |x| < c\}$.

Proof: $\sum_{n=0}^{\infty} a_n c^n$ converges. upper bound.

This means $|a_n|\,|c^n| \le M > 0$ $\forall n \in \mathbb{N} \cup \{0\}$.

Suppose we fix $x \in \mathbb{R}$ s.t. $|x| < c$.

So, what we have shown is, if the power series converges at some non-zero point on the real line; then in fact it converges in the interval, open interval $[-c, c]$, ok.

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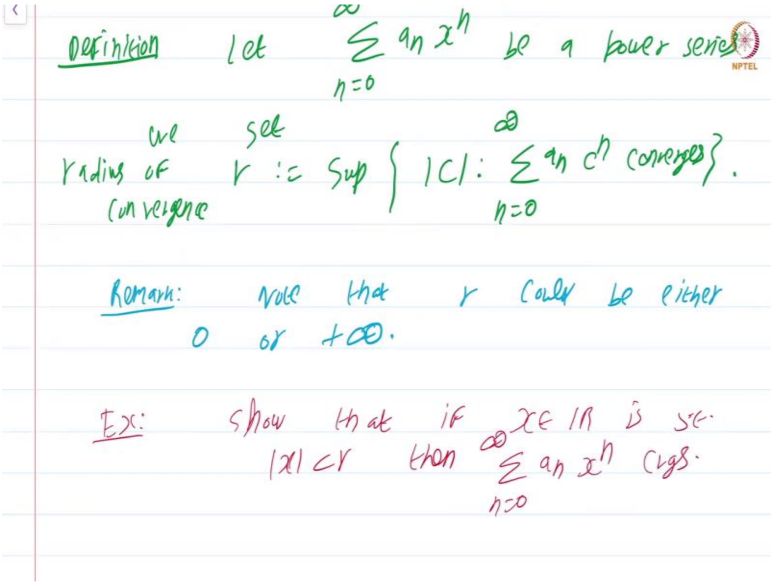

So, this prompts the following definition, this prompts the following definition, this prompts the following definition; let $\sum_{n=0}^{\infty} a_n\, x^n$ be a power series, we set r which is called the radius of convergence radius of convergence to be by definition, $r = sup\{\,|c|: \sum_{n=0}^{\infty} a_n\, c^n$converges$\}$, ok.

So, it is you take look at all the points where the series $\sum_{n=0}^{\infty} a_n\, c^n$ converges and take the supremum. Note that, so let me write it as a remark; note that r could be either 0 or ∞, both possibilities can happen, ok. Now, I am going to leave you with a very simple exercise; show that if $x\ \in R$ is such tha $|x| < r$ is, then $\sum_{n=0}^{\infty} a_n\, x^n$ converges absolutely.

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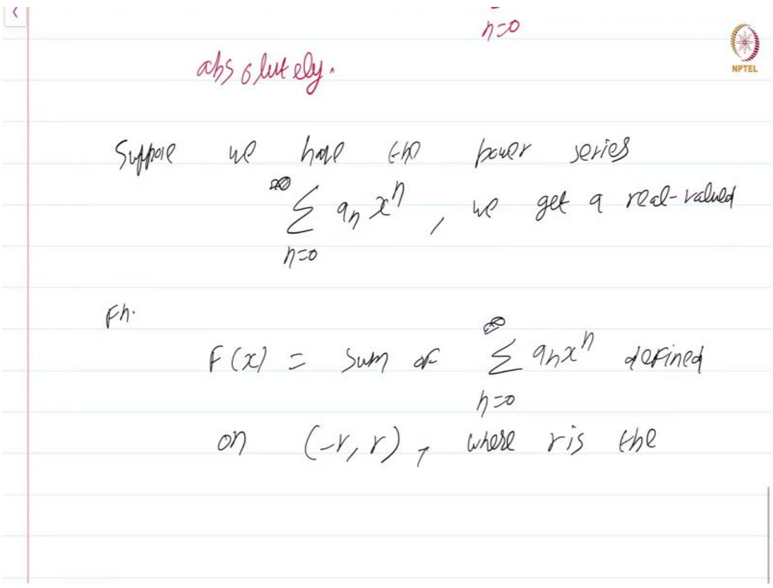

This is a very simple exercise, it follows immediately from the definition of this radius of convergence. And the fact that, whenever you have a point of convergence; then for all points whose modulus, whose absolute value is less than this given then $|c|$, then the series converges, essentially the previous proposition plus the definition of radius of convergence will immediately deliver the proof, ok.

So, where does this leave us? Suppose we have a series star we have the power series star; let me just write it out in full, suppose we have the power series $\sum_{n=0}^{\infty} a_n x^n$ we get a function, we get a real valued function $f(x)$ which is defined to be$\sum_{n=0}^{\infty} a_n x^n$ defined on $(-r, r)$, where r is the radius of convergence, ok. So, we end up with a function $f: (-r, r) \to R$.

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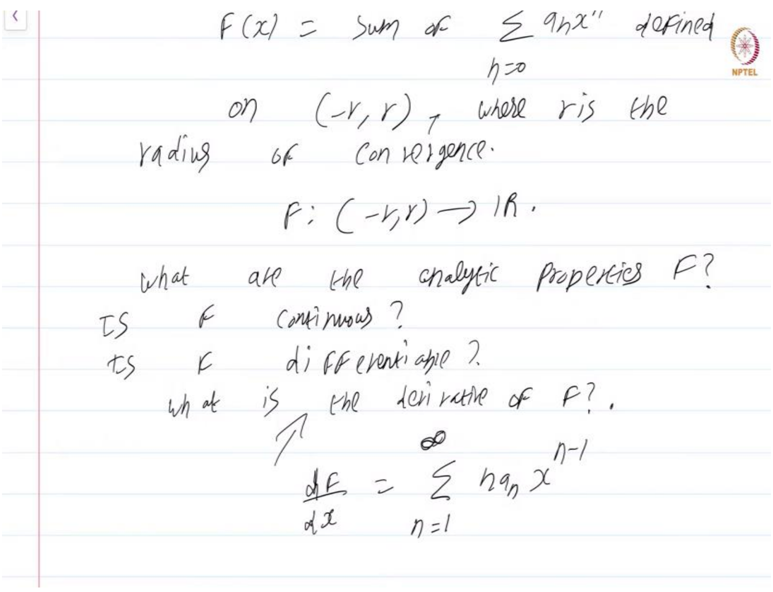

Now, the one crore question or the million dollar question is the following; what are the analytic properties of f? By analytic properties of f; I mean things that we have studied in analysis for instance is f continuous ok, is f differentiable ok? What is the derivative of f; if at all f is differentiable, what is the derivative of f, ok?

So, we have these following natural questions. And let me just leave you with an intriguing puzzle, which is sort of a partial answer to the last question without proof of course. It looks like summation; the derivative $\frac{df}{dx}$ would be nothing, but $\sum_{n=1}^{\infty} n\, a_n\, x^{n-1}$, ok. This is something that is natural to expect when you look at the power series $\sum_{n=0}^{\infty} a_n\, x^n$.

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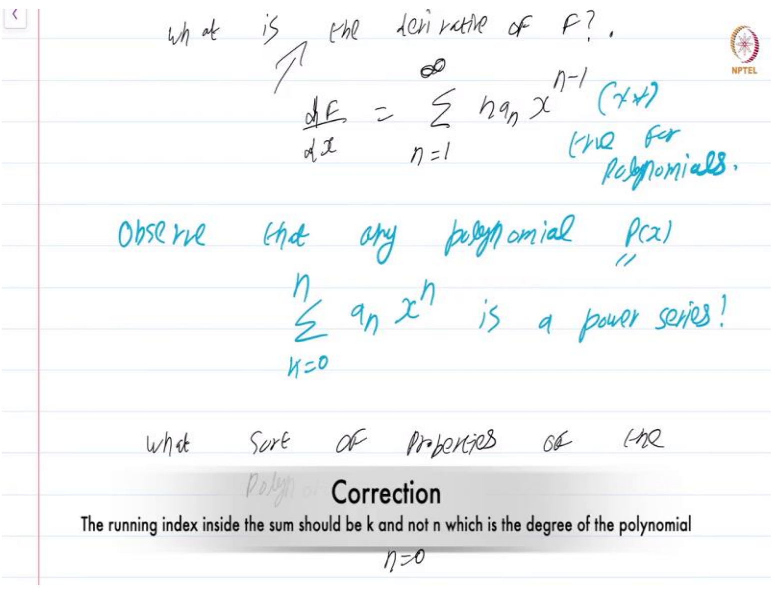

Why is it natural? Because observe that any polynomial $p(x)$, which is actually given by $\sum_{k=0}^{\mathrm{n}} a_k\, x^k$ is a power series, right. It is in fact a very nice power series, it is not an infinite power series; it is just the power series that terminates after a point. Note that in our definition of power series, we did not require all the coefficients to be non-zero or any such condition, a n's could be anything.

So, it just. So, happens that a polynomial is a special type of power series; a power series in which all, but finitely many coefficients are 0, ok. And for polynomials we have this result that, when you differentiate the monomial x^n, you get nx^{n-1}; therefore this formula double star is certainly true for polynomials, it is certainly true for polynomials.

So, you would expect this to be true for the power series also. In short what all this motivates is, what all this motivates is; what sort of properties of the polynomials, let us say some fixed n_0, $\sum_{n=0}^{n_0} a_n\, x^n$ migrate to the sum $\sum_{n=0}^{\infty} a_n\, x^n$.

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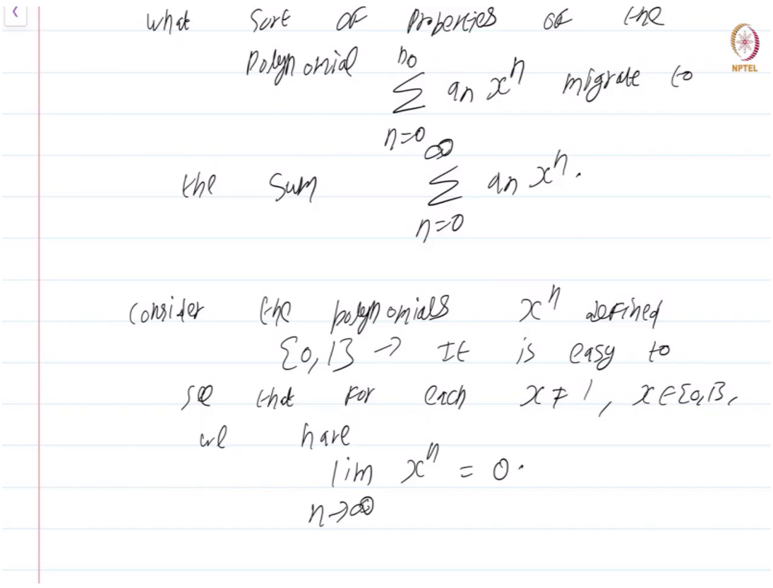

A power series is nothing, but a limit of polynomials; at least in the places where the series converges, it is sort of a limit of a polynomial, a sequence of polynomials. So, we are interested in seeing, we have polynomials which are about the nicest functions that you can take, many many properties are easy to see for polynomials; do these just migrate to the limit?

In this with relation to these remarks, let us consider an example. Consider, the polynomials x^n defined on the closed interval $[0,1]$, ok. Now, it is easy to see that, for each $x \neq 1$, $x \in [0,1]$, we have $\lim_{n\to\infty} x^n = 0$. But $\lim_{n\to\infty} 1^n = 1$.

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$$\lim_{n \to \infty} x^n = 0.$$

$$\lim_{n \to \infty} 1^n = 1.$$

The limit of the monomials x^n converges to the fn. $f(x)$

$$= \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

Which is NOT CONTINUOUS!.

So, this limit function; the limit of the monomials x^n converges to the function $f(x)$ which is defined as follows 0 if $x \in [0,1)$ and 1 if $x = 1$, which is not continuous. So, the limit of the very simple monomials themselves need not be a continuous function.

So, something weird is happening even when you take limits of very very simple polynomials. Now, the question is; can we put some hypothesis to ensure that the limit is continuous? And that is the next and the final major topic of this course uniform convergence. This is a course on Real Analysis and you have just watched the module on Power Series.