**Real Analysis - I**
**Dr. Jaikrishnan J**
**Department of Mathematics**
**Indian Institute of Technology, Palakkad**

**Lecture – 28.4**
**Smooth functions and Taylor Series**

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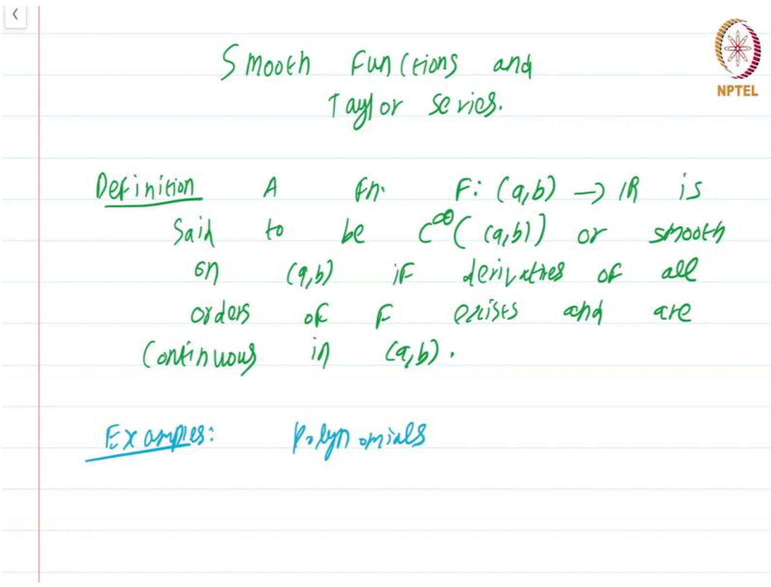


Let us begin by defining one of the most important classes of functions in all of analysis, in fact in all of mathematics. These are the functions that are extensively used both within mathematics and outside mathematics in applications to physics etcetera. This is the definition of a Smooth function.

Definition: A function $f: [a, b] \longrightarrow R$ is said to be $C^{\infty}([a, b])$ or just smooth on $[a, b]$; if all derivatives of all orders of $f$ exists and are continuous in $[a, b]$; that means, not only do the derivatives of $f$ exist first derivative, second derivative third derivative and so on, they are continuous as well..

Note that this continuity part is sort of redundant, simply because if the derivative $f^3$ exists at all points; then $f^2$, the second derivative is automatically continuous so on and so forth. We could have just said the derivatives of all orders exist at all points that would be enough. But I want to emphasize continuity also, so I am adding that ok. Now, examples ,there are plenty, since I said this is the most important.

Examples, polynomials are the good class of examples. Then, what else? Nothing, all the other functions that we have so far studied I have taken it for granted that you know. The polynomial functions are the most general class of functions we have studied so far.

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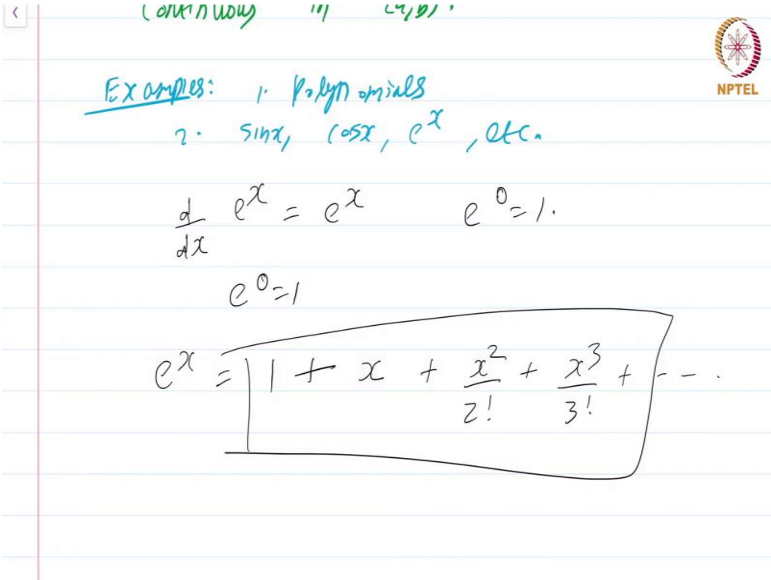


There are other examples. There are $\sin x$, $\cos x$, $e^x$ etcetera, the so called elementary functions; but so far, I have just taken it for granted that these functions are defined and have the properties that you are familiar with. We have so far not yet defined these functions precisely, which is sort of going against the goal of this course to define everything precisely.

So, how are we going to define these commonly used smooth functions? Well, one idea motivated from the previous theorem about Taylor, which I have sketched is why do not we just write down the Taylor series of $e^x$ or $\sin x$ or $\cos x$ and try to see what happens.

Well, we already know that the characteristic property of $e^x$ is that $\frac{d}{dx}e^x$ is equal to $e^x$ and $e^0 = 1$. Immediately, if you start writing down the first few terms in the Taylor series of this function $e^x$ at the point 0, you will get that $e^0$ which is 1.

So, $e^x = 1$ plus, then the first derivative of $e^x$ is just $e^x$ and at 0, it is 1. Then, you will just get $1 + x$ ok. Then, you will get $\frac{x^2}{2!}$ because again, the second derivative of $e^x$ is just $e^x$ and at 0, it is just 1, plus $\frac{x^3}{3!} + \ldots$

What I have done is, I have taken the Taylor's theorem that we have before that if you have a function that is $k+\ 1$ smooth, then you can write the function value at $x$ as the values of $f(0)$ $f'(0)$, $f''(0)$; that is essentially this Taylor polynomial plus a remainder term. What I am doing is, I am throwing away the remainder term and making this an infinite series.

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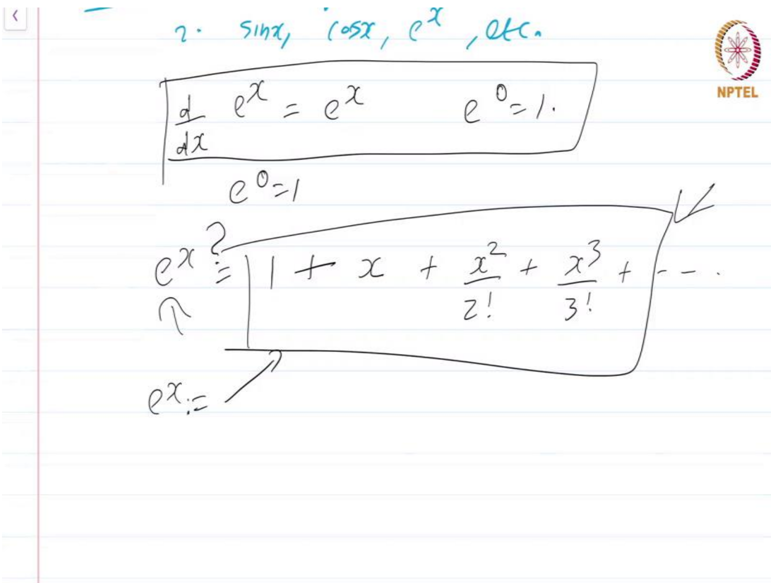


Now, question mark is which I should put a giant question mark here is this true. Is it really true that $e^x = 1 + x + \frac{x^2}{2!}+\frac{x^3}{3!}+...$? Ok. That is the question. If in fact, it is true; then, we can just reverse engineer the whole damn thing and just start by saying that $e^x$ is just this by definition ok.

We can just start our definition of $e^x$ is this and then, show that it has all the properties that characterize $e^x$; namely, these two ok. Now, that is the idea. Now, before that why would you even suspect that this right hand side, this infinite series is not equal to $e^x$; why would you even have a doubt in the first place?

And that doubt comes because of the existence of many many peculiar smooth functions and that is one of the most famous functions in more advanced mathematics which is extensively used in the theory of manifolds, this is called the Bump function.

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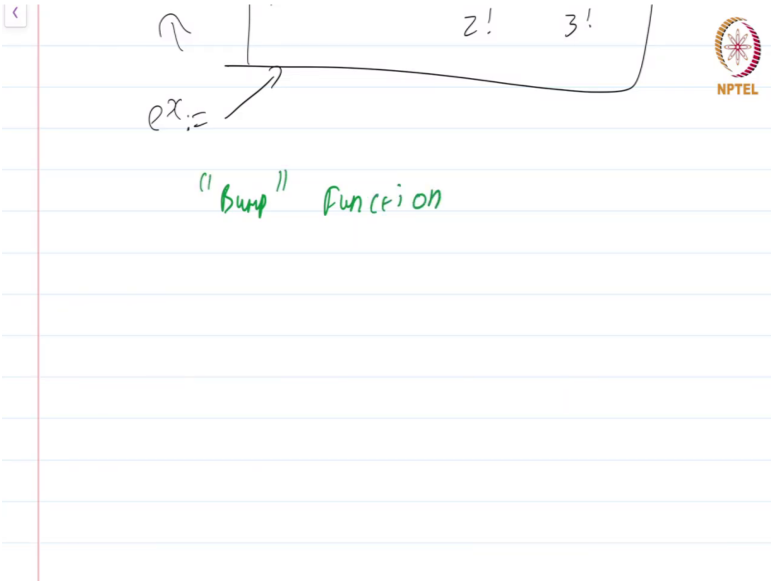


I mean this is not essentially the bump function. The bump function can be constructed using this function; but I am going to call it the bump function actually, better terminology would be Infinitely “flat” function.

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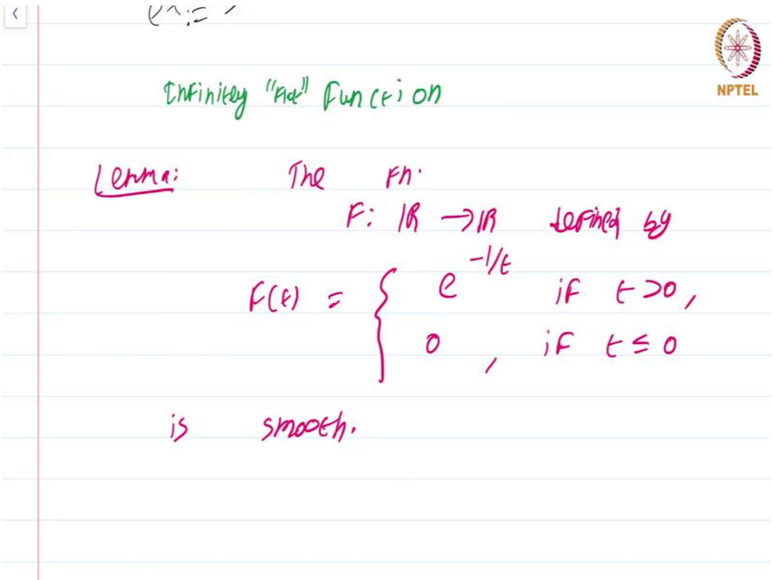


You will construct the bump function in the exercises using this infinitely flat function. How is this function defined? Well, let me just immediately begin with the lemma, which will have the definition in it.

Lemma, the function $f: R \longrightarrow R$ defined by defined by $f(t) = e^{\frac{-1}{t}}$ if $t > 0$ and $f(t) = 0$ if $t \leq 0$ is smooth ok. This function which is defined in two pieces as $e^{\frac{-1}{t}}$, if $t > 0$ and 0, if $t \leq 0$ is actually a smooth function.

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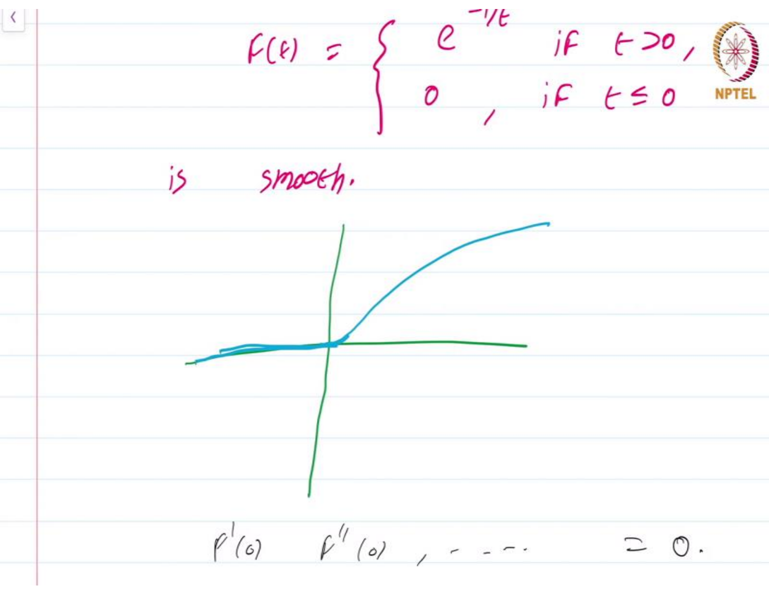


A picture, I urge you to draw a picture of this function will look something like this. In the negative axis, it is actually fully 0; its it is just fully 0, but it is sort of infinitely flat at the origin. It is sort of infinitely flat at the origin, it attaches to this straight line that is nothing but the negative $x$ axis in a smooth way. So, this function sort of becomes infinitely flat as you approach the origin from the right ok.

Now, what is this function trying to tell us? Well, if you were to write down the Taylor series of this function, the infinite Taylor series you are in for a shock because $f'(0)$, $f''(0)$ and so on and so forth are all 0, all the coefficients in this Taylor expansion will be 0.

So, if you were to write down the Taylor series of this function, you would get a giant 0; which is obviously not equal $e^{\frac{-1}{t}}$ which is what the function is when t is greater than 0. So, the Taylor series of a smooth function might not have anything to do with the function.

We are getting a giant 0 here; whereas, the function is $e^{\frac{-1}{t}}$, when t > 0. So, it is not always true that the Taylor series, the infinite Taylor series that you just write down formally will agree with the function, that might not always happen. Let us first see a proof that this function is in fact smooth ok.

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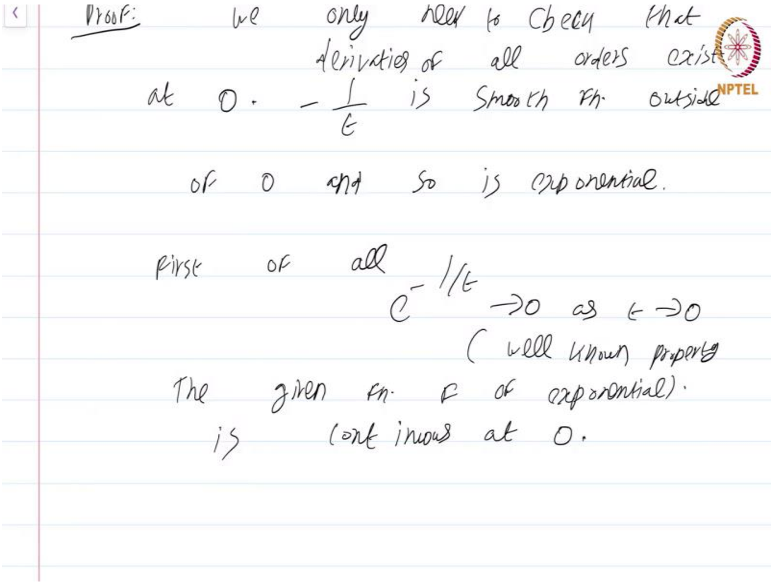


So, all we have to do is that we only need to check that all derivatives or better yet derivatives of all orders derivatives exist at 0 ok. The reason is that $\frac{1}{x}$ is a continuous function; $\frac{1}{x}$ is in fact smooth function outside of 0 and so is exponential.

So, is exponential. Let me just write $\frac{1}{t}$; or rather in our case $\frac{-1}{t}$ and so, is exponential. Therefore, $e^{\frac{-1}{t}}$; which is going to be a composition of smooth functions will be smooth which you can prove by induction.

It is an easy argument to show that $e^{\frac{-1}{t}}$ will be smooth, everywhere except the origin. The origin is the only problematic point, where two different functions are being joined. Of course, what I said about $e^{\frac{-1}{t}}$, applies only to $t > 0$.

for $t < 0$, you do not have to do any work; it is just the identically constant function 0. So, the only problematic point is the origin is the origin. First of all $e^{\frac{-1}{t}}$ converges to 0 as t goes to 0. This is just a well-known property of the exponential function ok.

This means, the given function $f$ is continuous at 0. At least, we are off to a good start, continuity follows easily ok. Now, we have to show that the function $f$ is actually differentiable at the origin. We have to show that it is differentiable at the origin. If at all the function is going to be differentiable at the origin, the derivative has to be 0. That is very clear because the function is identically 0, whenever $t \leq 0$ ok.

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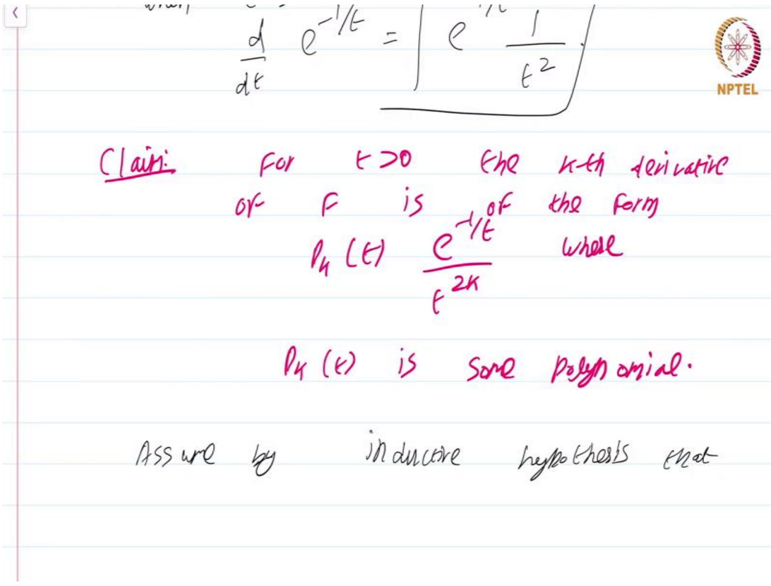


So, when $t > 0$, $\frac{d}{dt} e^{\frac{-1}{t}} = e^{\frac{-1}{t}} \frac{d}{dt} \frac{1}{t} = e^{\frac{-1}{t}} \frac{1}{t^2}$, ok. We now have the following claim for $t > 0$, the kth derivative, kth derivative of $f$ is of the form of the form $P_k(t) e^{\frac{-1}{t}} \frac{1}{t^{2k}}$, where $P_k(t)$ is some polynomial some polynomial.

We have already dealt with the base case that is what $\frac{d}{dt} e^{\frac{-1}{t}} = e^{\frac{-1}{t}} \frac{1}{t^2}$ is essentially is same. What I am saying now is that this is true for all derivatives. So, assume by inductive hypothesis that $f^{(n)}(t)$ is actually of the form $f^{(n)}(t) = P_n(t)\ e^{\frac{-1}{t}} \frac{1}{t^{2n}}$.

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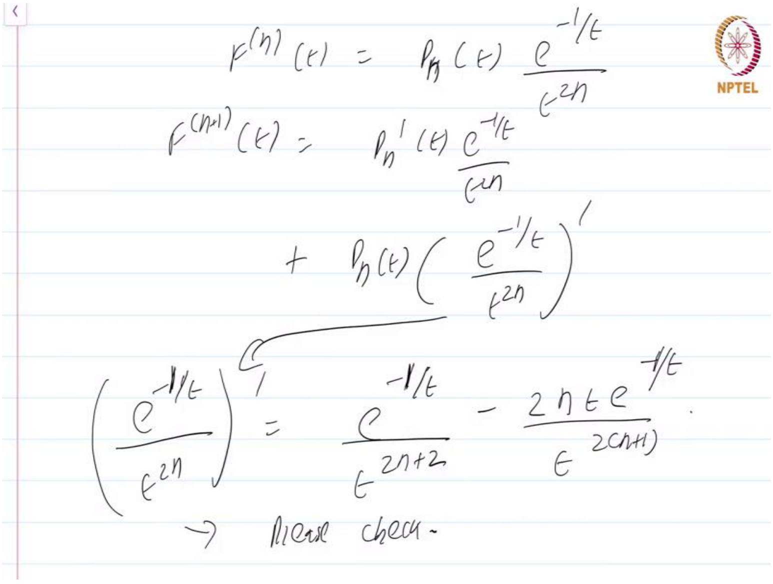


Now, we will have to show the same thing for the $(n+1)$th derivative. So, $f^{n+1}(t) = P_n{}'(t)$ $e^{\frac{-1}{t}}\frac{1}{t^{2n}}$, I am applying the product rule first, plus $P_n(t)\left(e^{\frac{-1}{t}}\frac{1}{t^{2n}}\right)'$ , ok.

Now, how do you differentiate this? This is just going to be, I will just concentrate on this term $e^{\frac{-1}{t}}\frac{1}{t^{2n}}$. The derivative of this, this is going to be nothing but $e^{\frac{-1}{t}}\frac{1}{t^{2n+2}}$, I am essentially differentiating the numerator which is $e^{\frac{-1}{t}}$which is going to give $-\frac{1}{t^2}$ squared and another minus sign, both will get cancelled. So, I will get $\frac{1}{t^2}$. This $\frac{1}{t^2}$, I am attaching to the denominator plus the derivative of $\frac{1}{t^{2n}}$ ok.

That is going to be minu,s it is going to be $\frac{e^{-1/t}}{t^{2n+2}} - \frac{2nte^{-1/t}}{t^{2(n+1)}}$. So, please check this; please check ok.

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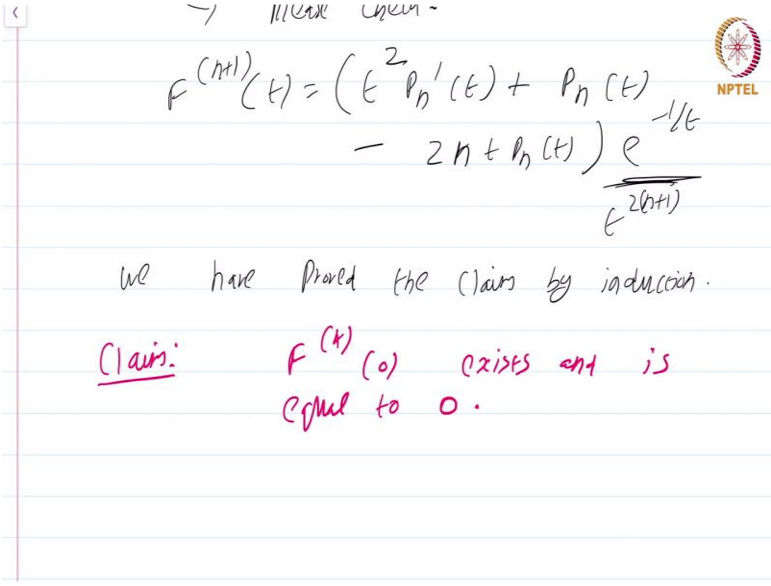


So, I have just differentiated $\frac{1}{t^{2n}}$ and we get this ok. Now, what does all this give us? It is going to give that $f^{n+1}(t) = t^2 P_n'(t) + P_n(t) - 2ntP_n(t)\frac{e^{-1/t}}{t^{2(n+1)}}$ ok. So, this is just a routine computation.

Please check or check the notes for all the details worked out ok. So, we have proved this; we have proved the claim by induction. So, now, we exactly know how $f^{n+1}(t)$, at $t > 0$ is going to look like ok.

Now, what is our aim is another claim which is going to be proved again by induction. Claim is that $f^k(0)$ exists and is equal to 0 which is what we want to show which is equal to 0 ok. Obviously, if $f^k(0)$ exists, it is got to be equal to 0 because the left hand derivative, when you take just derivative from the left it is going to be 0 because that function is just identically 0, when $t < 0$ ok.

So, what we have to do is to show that when you take the derivative from the right, you still get 0 for this to work ok.

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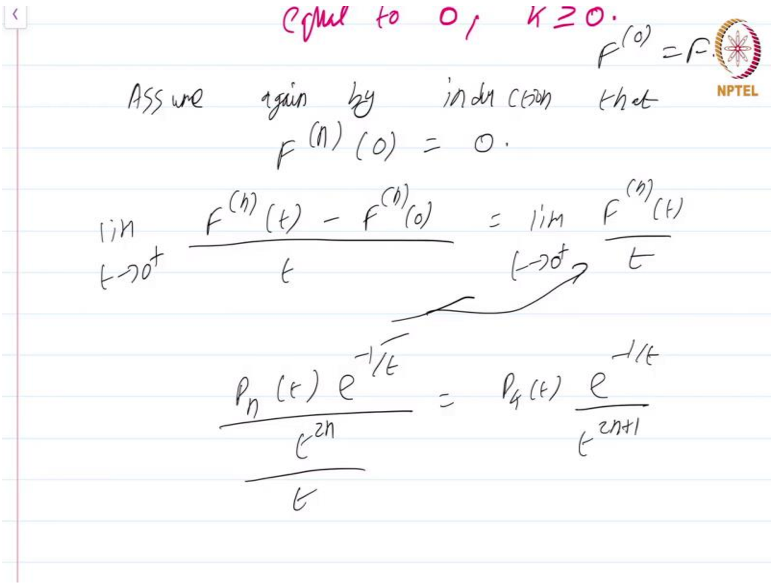


So, assume again by induction that f k of 0 is equal to 0; f n $f^n(0) = 0$. Assume that you have shown this ok. So, let me just make a remark if this is; if this is confusing, this is for k greater than or equal to 0 ok. We have already shown that $f(0) = 0$.

So, the base case is essentially done. Recall that when you put $f^0$, you just mean $f$. When you put $f^0$, f 0 is just another thing for $f$ ok. So, you have you are taking $f^n(0)$ and we want to show that it is going to be; we are assuming that it is going to be 0 and we want to compute the $(n+1)$th derivative of $f$ at the origin.

Again, what you do is this is nothing but $\lim_{t \to 0^+} \frac{f^n(t) - f^n(0)}{t}$, this is nothing but the definition of the derivative from the right at the origin. We already know that $f^n(0) = 0$ by induction hypothesis. So, this is just $\lim_{t \to 0^+} \frac{f^n(t)}{t}$.

Now, you might understand why we try to claim that $f^n(t)$is of a particular form. We know that $f^n(t)$this is going to be of a particular form. What is$f^n(t)$? This is just $\frac{\frac{P_n(t)e^{\frac{-1}{t}}}{t^{2n}}}{t}$, that is what the expression inside is going to be ok. So, this is just $\frac{P_k(t)e^{\frac{-1}{t}}}{t^{2n+1}}$, ok.

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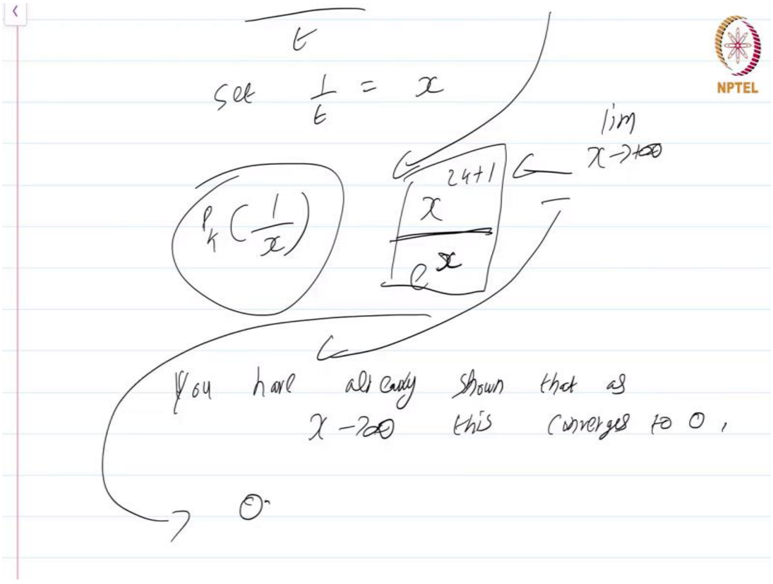


Now, set $\frac{1}{t} = x$. So, this entire expression will just become $P_k\left(\frac{1}{x}\right)\frac{x^{2k+1}}{e^x}$ and now we have to compute limit $x$ going to plus infinity of this.

We have to compute limit $x$ going to plus infinity of this term ok. But this was already given as an exercise. You have already solved; you have already shown that as $x$ goes to infinity, this quantity, this this converges to 0, ok.

This is one of the exercises which was first given just for sequences in the chapter on sequences. Again, for functions in the chapter on continuity and limits ok. So, this this first term goes to 0 and this polynomial as $x$ approaches infinity, $\frac{1}{x}$ approaches 0. So, this is going to go to some fixed constant. So, this whole thing converges to 0; this whole thing converges to 0.

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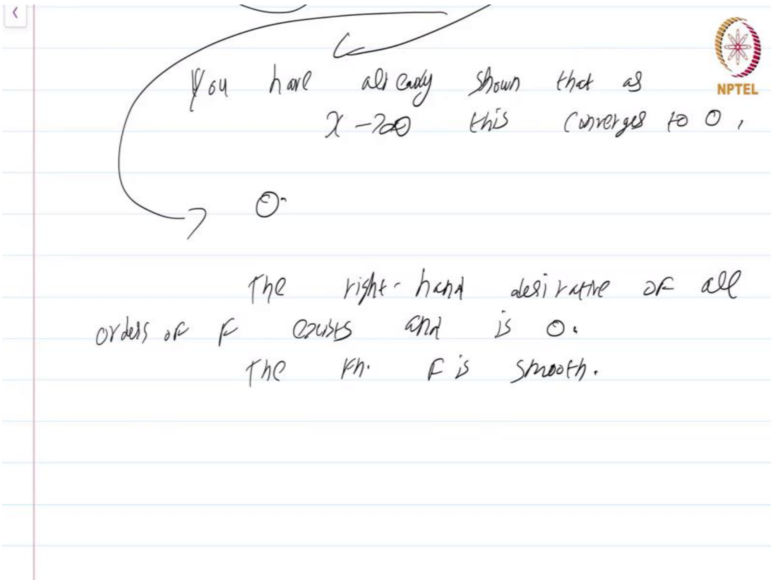


The net upshot of all this is that the right hand derivative of $f$ exists of all orders exists and is 0. So, the net upshot is this is a smooth function; the function $f$ is smooth is smooth.

Yet all derivatives of $f$ at the origin are just 0 which means when you write down the Taylor series, you will get a giant 0. So, it is not true that if you have a smooth function and you write down its Taylor series, infinite Taylor series you get anything useful. It is not always the case that you will get anything useful.

So, now we are going to begin the topic of power series, we are going to study the general situation under which it happens that the Taylor series does in fact converge to the given function.

Then, we are going to reverse engineer and define exponential and trigonometric sum trigonometric functions directly using these power series and prove that they have the basic properties that they indeed have we have taken it for granted whatever properties, we have taken it for granted, we are going to prove that they indeed have those properties.

This is a course on Real Analysis and you have just watched the module on Smooth functions and Taylor Series.