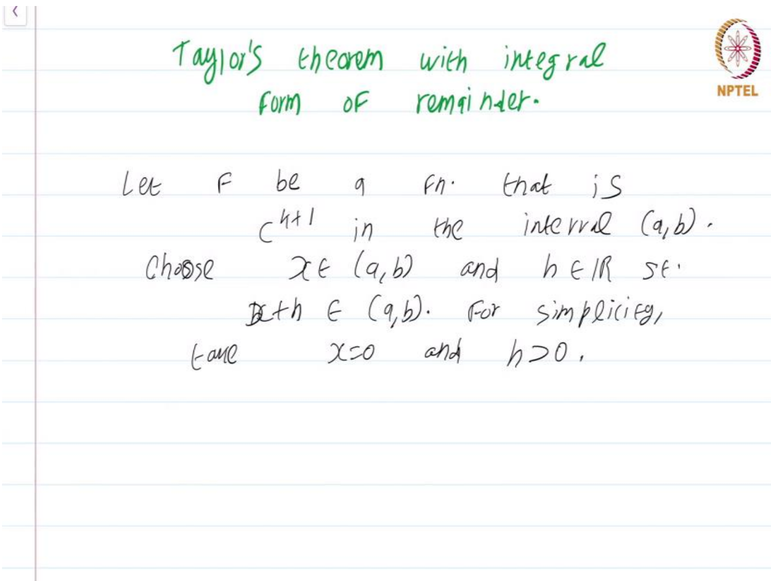
Real Analysis - I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 28.2
Taylor's Theorem: Integral form of Remainder

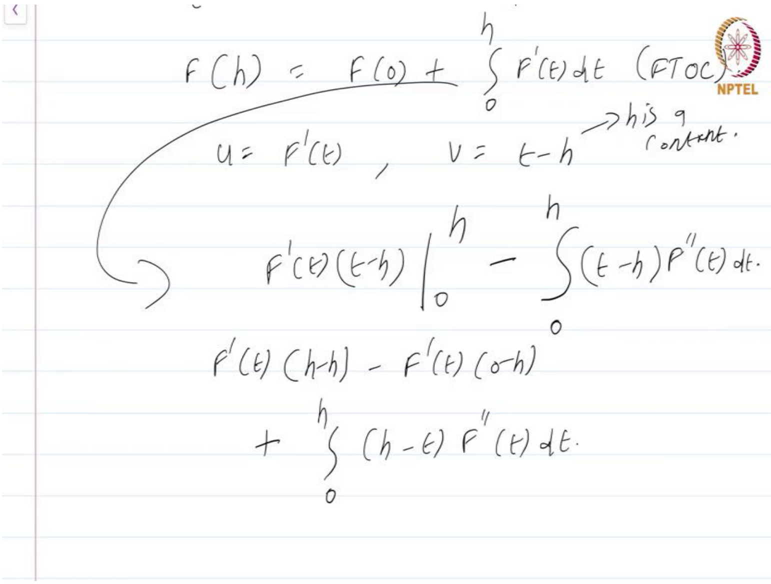
(Refer Slide Time: 00:13)



Now, that we have the powerful fundamental theorem of calculus and integration by parts in our toolkit; let us apply this to get another form of the reminder term in Taylor's theorem. So, what I will do is, I will just sketch the argument and leave it to you to formulate a precise statement of Taylor's theorem with the integral form of reminder in the exercises.

So, let us begin with the hypothesis. So, let f be a function that is C^{k+1} in the interval $[a, b]$, ok. Now, choose $x \in [a, b]$ and $h \in R$, such that $x + h \in [a, b]$, ok. for simplicity, this is the sketchy part; for simplicity, take $x = 0$ and $h > 0$. This is not really; I mean the argument that is about to follow does not crucially depend on these hypothesis, but the notation will become significantly simpler, ok.

(Refer Slide Time: 01:54)



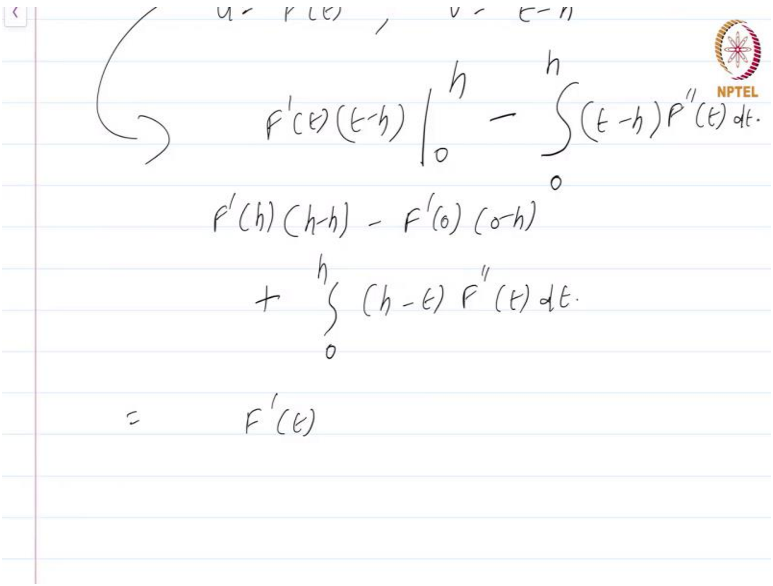
Now, what we are about to do is to derive the value of f, the value of f at the point h in terms of the value of f at the point 0, that is the aim. Now, first of all the fundamental theorem of calculus immediately gives that $f(h) = f(0) + \int_0^h f(t)dt$, right. This is just the fundamental theorem of calculus. Now, I am going to apply the fundamental theorem of calculus yet again to get the expression $\int_0^h f\,'(t)dt$ into a new form.

What I am going to do is, I am going to first revert back using a time machine back to high school notation; I am going to take $u = f'(t)$ and $v = t - h$, note h is a constant.

Now, integration by parts gives that, this is nothing but u, v, which is $f'(t)t - h|_0^h$ - $\int_0^h (t-h)f\,''(t)dt$, ok. This is just your standard integration by parts, even applied with the standard notation that you are familiar with from high school. Now, let us see what happens.

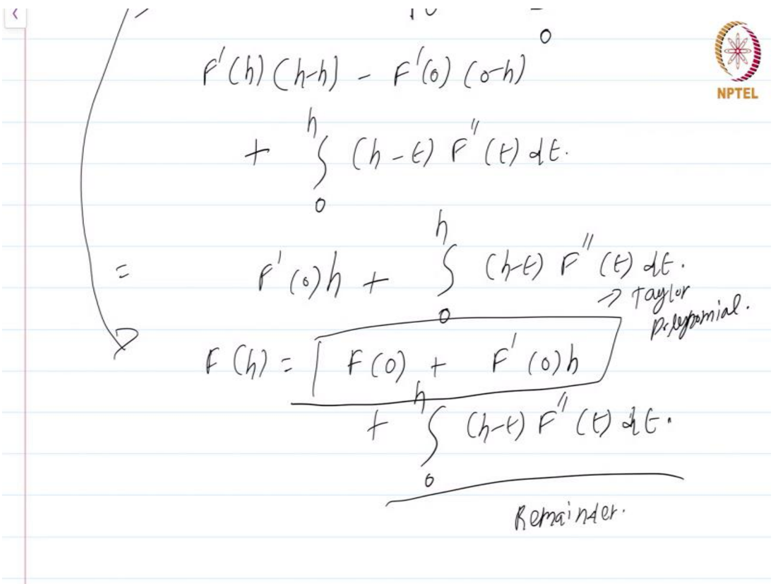
This you will get the first term; you will get $f'(h)(h-h) - f'(0)(0-h)$. And if you do not mind, this – sign. I will make it a + sign and write $\int_0^h (h-t)f\,''(t)dt$. So, I have done nothing, I have just taken the $-$ sign inside.

(Refer Slide Time: 04:14)



Now, observe that there is a $-$ sign near $-f'(t)(0-h)$and there is a $-$ sign inside. So, this whole thing just becomes $f'(t)h$, sorry $f'(0)$, sorry about that, this should be $f'(h)$ and this should be $f'(0)$. I forgot to substitute the limits for f'.

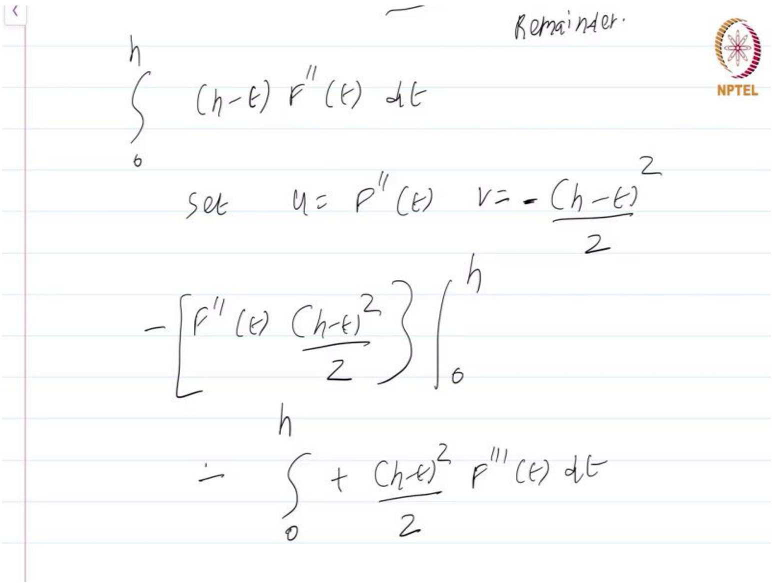
(Refer Slide Time: 04:36)



So, this will just give you $f'(0) + \int_0^h (h-t) f''(t)dt$, so far so good. So, the original expression, the original expression which I bring back here will just give us; wait a second, I just forgot, I forgot to put a h here, ok. So, you will get $f(h) = f(0) + f'(0)h + \int_0^h (h-t) f''(t)dt$, ok.

So, we have got some sort of approximation of $f(h)$. Note that this is the this is a Taylor polynomial, this is a Taylor polynomial, ok. And this is the reminder, this is the reminder.

(Refer Slide Time: 05:48)

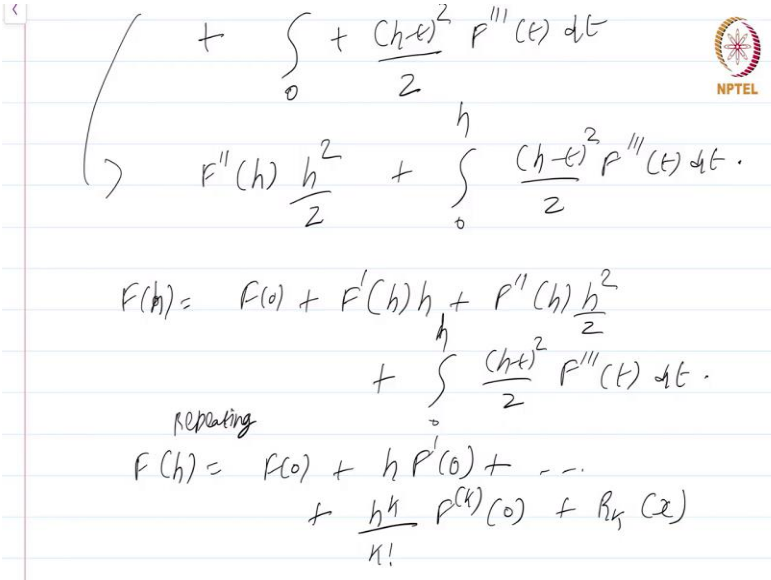


Now, what do you do? Well, we can repeat this argument one other step; look at $\int_0^h (h-t) f''(t) dt$. Look at this term, ok. Now, what you do is, set $u = f''(t)$ and $v = \frac{-(h-t)^2}{2}$. In fact, put a negative sign, put a negative sign; simply because when you differentiate $\frac{(h-t)^2}{2}$, you will get $2(h-t) \cdot -1$. So, I want to get rid of that, ok.

Now, again apply integration by parts what you will get is, uv, that is $-\left[\frac{f''(t)(h-t)^2}{2} |_0^h\right]$ - $\int_0^h v\, du$, which is, $-\int_0^h -\frac{(h-t)^2}{2} f'''(t) dt$ t and these two $-$ signs just go away on its own, no need to do any additional work.

So, what about the first term? We will have to substitute the limits again. So, you will get, you will get that when you substitute h, you it will disappear; when you substitute 0, there is again a negative sign and that will get cancelled with the negative sign outside.

(Refer Slide Time: 07:31)

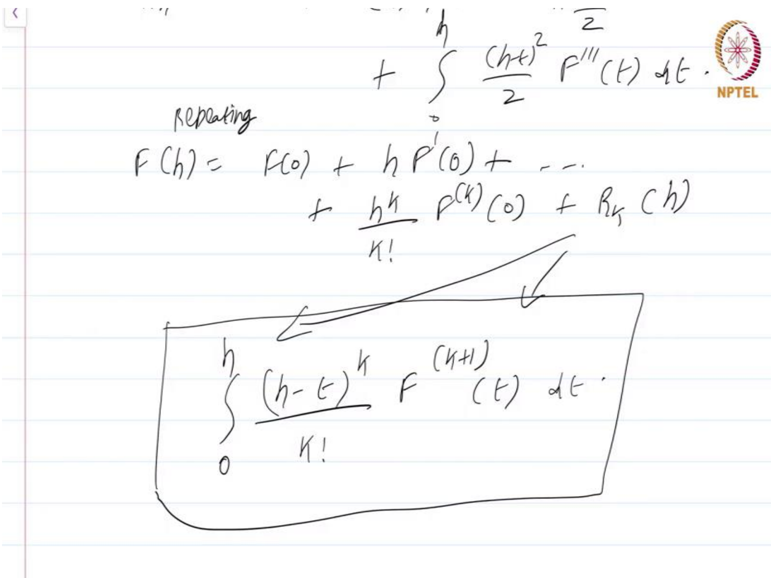


So, this whole thing will become $f''(h)\frac{h^2}{2} + \int_0^h \frac{(h-t)^2}{2} f'''(t)dt$. So, the original expression will become $f(h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2} + \int_0^h \frac{(h-t)^2}{2} f'''(t)dt$. Excellent, excellent.

Repeating this argument repeating this argument, you can get that $f(h)$ is. So, repeating

$$f(h) = f(0) + f'(0)h + \cdots + \frac{h^k}{k!} f^{(k)}(0) + R_k(h) .$$

(Refer Slide Time: 09:05)



And what is this $R_k(x)$? Sorry $R_k(h)$, it is a function of h, $R_k(h)$. And what is this $R_k(h)$? It is $\int_0^h \frac{(h-t)^k}{k!} f^{k+1}(t)dt$. So, we have got a different form of the reminder by repeatedly applying integration by parts, in a sense by repeatedly applying the fundamental theorem of calculus.

Now, you might ask that this application of integration by parts to get this reminder term, seems like a lot of effort ugly computation; I mean, I personally think this argument is quite beautiful, but as you note as I myself made several simple mistakes.

So, working this out in detail, you might think it is ugly, why to do that, the previous argument was far more transparent what not; the remainder term which involved just repeated application of mean value theorem, which we have seen before.

Now, there is a big advantage of this particular expression; the big advantage is that, there is no unknown point in between 0 and h that is involved in the remainder term. If you recall that the earlier Lagrange form of the reminder that we had got involved an unknown quantity which we cannot possibly determine, right. Whereas, this is an integral which it is more tractable.

for instance, a simple example; suppose you are trying to write down the series for sin x or cos x, which we will do just a few modules down the line. Observe that we already know that the derivative of sin x is cos x and the derivative of $cos\ x$ is $-\ sin\ x$.

So, in some sense we actually know what this $f^{k+1}(t)$ is going to be in that scenario; we know the exact function and it might be useful to have an expression like this. Whereas, the Lagrange form of the reminder, when you are trying to write down the series for $sin\ x$ or $cos\ x$, will involve a term which cannot possibly be fully determined; because it will involve a point in between 0 and h.

Now, both forms have their advantages and disadvantages, which we will explore in great detail when we start studying power series and try to show convergence of certain power series involving sine, cosine and so on. This is a course on real analysis and you have just watched the module on Taylor's series with integral form of the reminder.