Real Analysis - I
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Lecture – 28.1
The fundamental Theorem of Calculus

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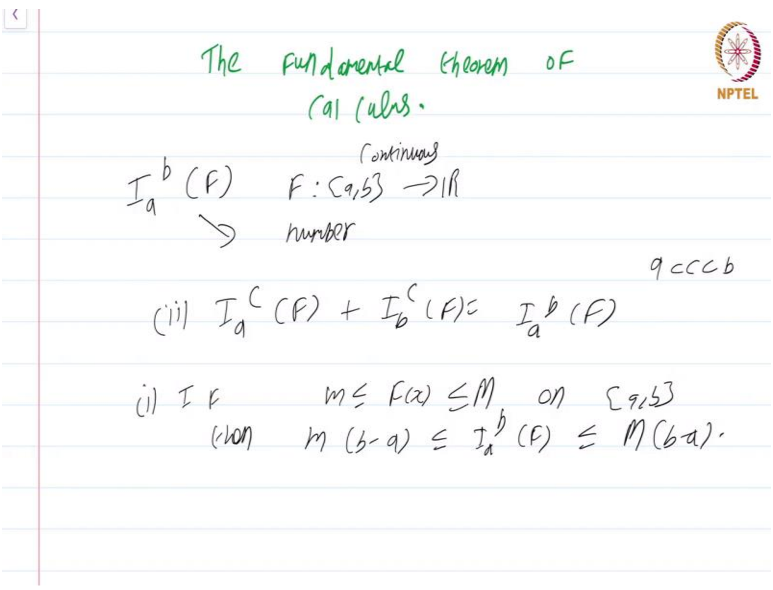

Now that we have established several powerful properties of the Riemann integral we can tie it back to the axiomatic characterization that we saw way back in the very first module in the series on Riemann integration. So, recall that in the very first module we had considered a function of the form $I_a^b(f)$ that associates with a function.

So, $f:[a,b]\longrightarrow R$ is a continuous function and $I_a^b(f)$ is sort of a function defined on continuous functions that assigns a number; that assigns a number and this number satisfies these two properties that $I_a^c(f)+I_b^c(f)$ = $I_a^b(f)$, where $a<c<\ b$. This is one of the properties.

And, the second property is that or rather this was the second property. So, the first property was that if $m\leq f(x)\leq M$ on this closed interval $[a,b]$, then $m(b-a)\leq I_a^b(f)\leq M(b-a)$.

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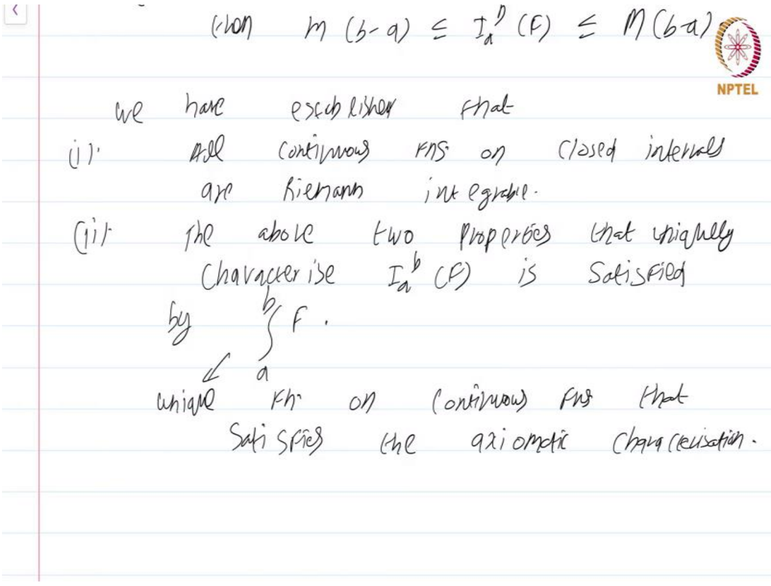

Now, we have established that first of all, all continuous functions on closed intervals are Riemann integrable. We have already established this.

Then we have also established that the above two properties; the above two properties that uniquely characterize $I_a^b(f)$ is satisfied by $\int_a^b f$. So, that means, that this is the unique function this is the unique function on continuous functions that satisfies the axiomatic characterization; the axiomatic characterization.

So, this says that the function I_a^b whose existence was up in the air is now no longer up in the air its on concrete. We have created the appropriate definitions that will make I_a^b exist and this existence is shown to be nothing, but the Riemann integral that we have been studying. Now, this helps us a lot because the topic of this module is the fundamental theorem of calculus, we have already done the hard work.

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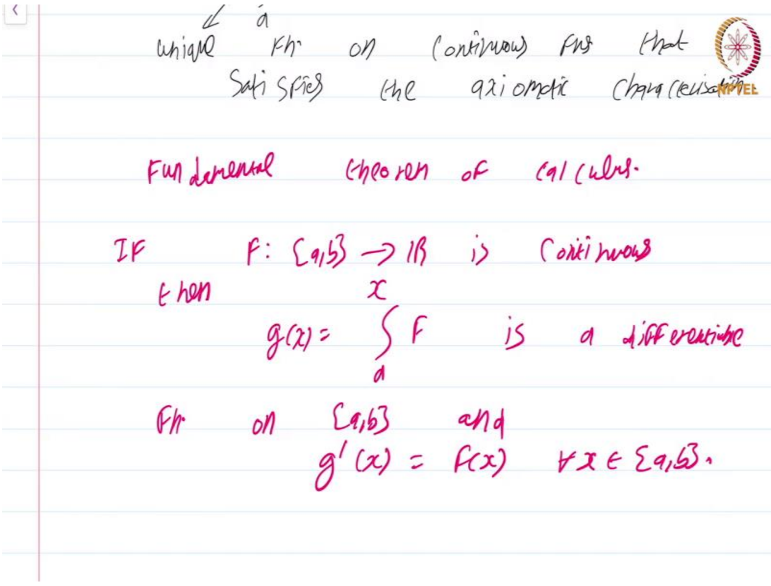

So, I can state the fundamental theorem of now. Fundamental Theorem of calculus: if $f: [a, b] \longrightarrow R$ is Riemann integrable or is continuous I got a head of myself. The version I was about to state is going to be left to you in the exercises. $f: [a, b] \longrightarrow R$ is continuous then $g(x) = \int_a^x f$ is a differentiable function on closed interval $[a, b]$ and $g'(x) = f(x)$ for all $x \in [a, b]$

This follows immediately because we have already proved this, that any I_a^b that satisfies that satisfies these two conditions is automatically going to satisfy the conclusion of the fundamental theorem of calculus, we have already seen that. So, this bit is done with zero effort.

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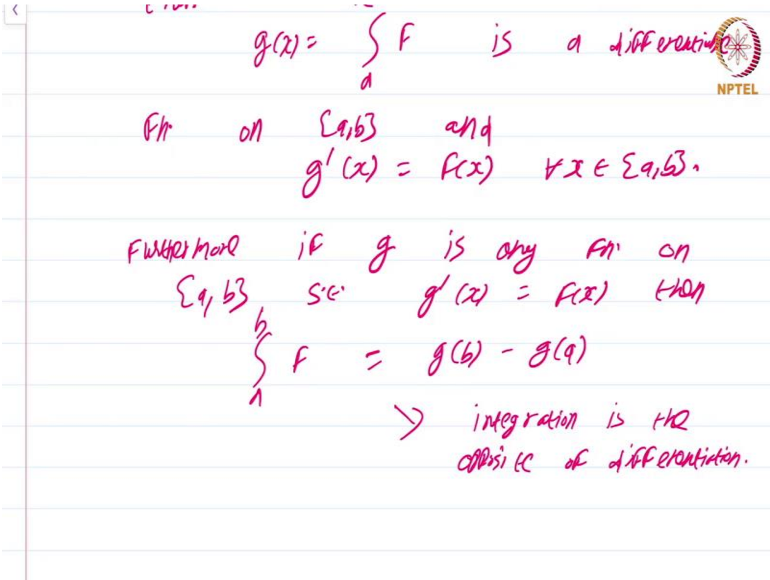

Furthermore, if g is any function on closed interval $[a, b]$ such that $g'(x) = f(x)$, then $\int_a^b f(x) = g(b) - g(a)$, ok. So, what this says is integration is the opposite of differentiation. So, if you start with a continuous function take any anti-derivative that is a function g such that $g'(x) = f(x)$,, then integrating a to b f is reduced to finding the function g and then just substituting the limits to get $g(b) - g(a)$, ok.

So, because we have that integration and differentiation or opposites several useful things like integration by substitution etcetera can be justified now. You have never formally seen a justification of these facts, but we can do them now.

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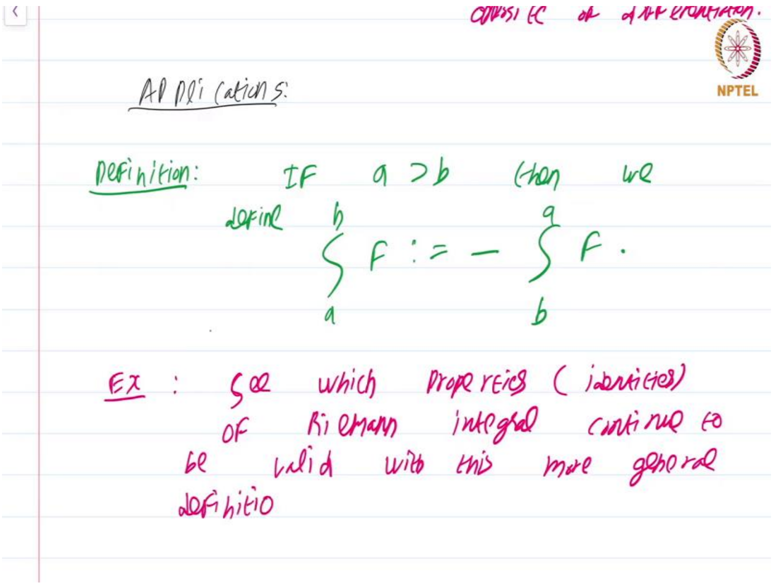

So, let us call these applications. By the way I should remark the way I have stated the fundamental theorem of calculus. I have stated it only for continuous functions even though you can say a lot more which is there as part of the exercises.

Since the crux the idea of the proof is there already in the axiomatic characterization and all you have to do is adapt the proofs and modify it little bit I am just stating it in this case in any scenario this is the most important version of the fundamental theorem of calculus. So, I am leaving it here ok.

So, in the exercises please look at modified statements of the fundamental theorem of calculus which applies to Riemann integrable functions not just continuous functions. Before we begin with the first application let me just give a definition let me just give a definition if $a > b$, then we define $\int_a^b f(x) = -\int_b^a f(x)$, this is the usual definition.

So, exercise which is not that such a difficult exercise see which properties. So, let me just put by definition equal to see which properties or rather identities of Riemann integral continue to be valid; continue to be valid with this more general definition; with this more general definition with this definition out of the way let us see some applications.

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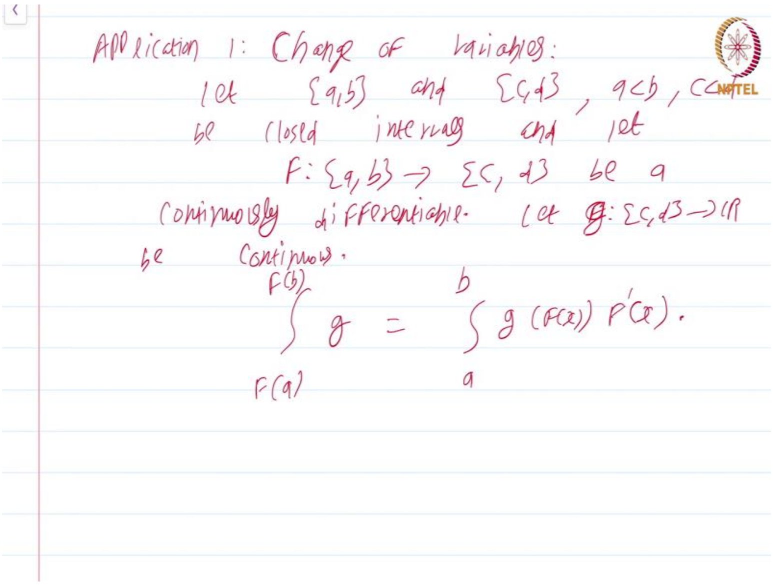

So, application 1. This application is also called the change of variables; change of variables. So, this states the following. Let $[a, b]$ and $[c, d]$ with $a < b, c < d$ be closed intervals be closed intervals and let $f: [a, b] \longrightarrow [c, d]$ be a continuously differentiable function.

Recall this just means that the function is differentiable and the derivative f' is continuous. Let $g: [c, d] \longrightarrow R$ be continuous. Now, the question that this change of variables seeks to answer is $\int_{f(a)}^{f(b)} g$, what is this going to be?

Note there is no necessity for $f(a)$ to be less than $f(b)$ it could be the case that $f(b)$ is actually less than $f(a)$. That is why I had to make the definition that I just made before I started with this application. So, $\int_{f(a)}^{f(b)} g$, as you all know is nothing, but $\int_a^b g\big(f(x)\big)f\,'(x)$, ok.

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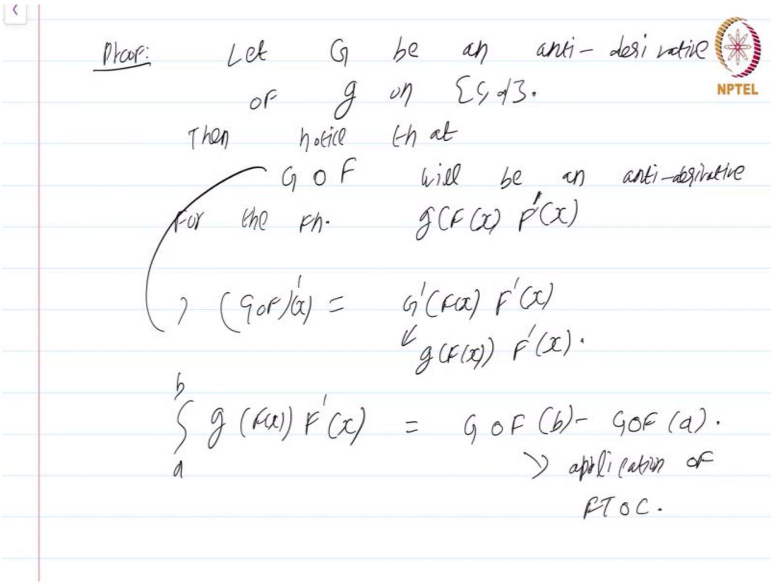

So, why is this true? Let us see a proof; let us see a proof. Now, let capital G be an anti derivative of g on $[c, d]$ such an anti derivative exists because the function g is given to be continuous. Then notice that; notice that; notice that $G \circ f$, G will be an anti derivative for the function; for the function $g\big(f(x)\big)f'(x)$.

This is just chain rule because $(G \circ f)'(x) = G'^{(f(x))}f'(x)$ and $G'\big(f(x)\big)f'(x)$ = $g\big(f(x)\big)f'(x)$. So, $G \circ f$ is an anti derivative for $g\big(f(x)\big)f'(x)$. What this says, I completely reverse the thing sorry about that, $\int_a^b g\big(f(x)\big)f\,'(x) = G \circ f(b) - G \circ f(a)$

So, this is just an application of the fundamental theorem of calculus is just an application of the fundamental theorem of calculus. Please look back in the original modules on the axiomatic characterization to see why this follows from the fundamental theorem of calculus immediately or better yet just sit down right now and prove it. Its just going to take you 2 minutes.

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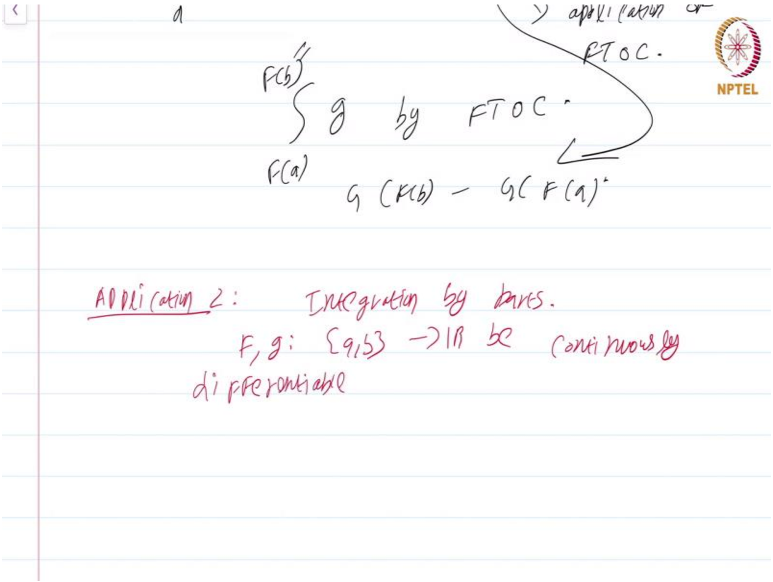

So $\int_a^b g(f(x))f'(x) = G(f(b)) - G(f(a)) = \int_{f(a)}^{f(b)} g$. Why is this the case? Well, again by fundamental theorem of calculus; by fundamental theorem of calculus because $\int_{f(a)}^{f(b)} g$ will be just capital $G(f(b)) - G(f(a))$ and this is same as this. This is same as this ok.

So, this is the famous integration by substitution, change of variables. There are several names for this integration by substitution or change of variables and the proof just follows immediately from the fundamental theorem of calculus. That is application 1. Let us see application 2.

Application 2 is one of the most powerful methods of integration that you have no doubt extensively studied in high school, this is called integration by parts; integration by parts. What does integration by parts say? Let $f, g: [a, b] \longrightarrow R$ continuously differentiable. So, take two functions f, g on the interval $[a, b]$ that are continuously differentiable.

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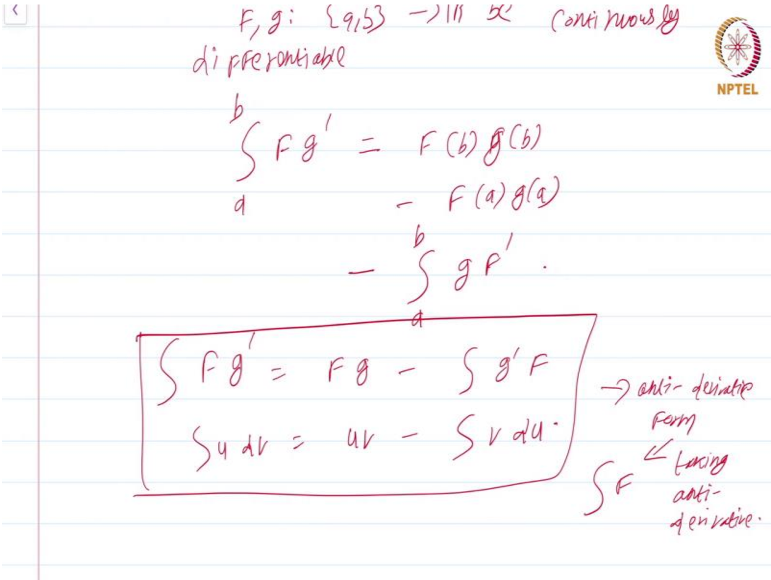

What you are interested in is $\int_a^b fg'$, ok. As we all know the answer to this is nothing, but $f(b)g(b) - f(a)g(a) - \int_a^b gf'$, you just reverse the derivatives that is it becomes gf', ok. You are all familiar with this $\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b gf'$.

If you are looking at this and thinking that it is different from what you saw in high school the only reason is because in high school you would have seen it in this form $\int_a^b fg' = fg - \int_a^b gf'$. You would have seen it in this form or $\int u\, dv = uv - \int v\, du$ ok.

Now, these are just; these are just anti-derivatives. These are the anti derivative form; the anti derivative form, where I am treating integral of f as meaning taking anti derivative; as meaning to take anti derivative. That is the only reason why this form might look the form that I have stated $\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b gf'$, that might look slightly different.

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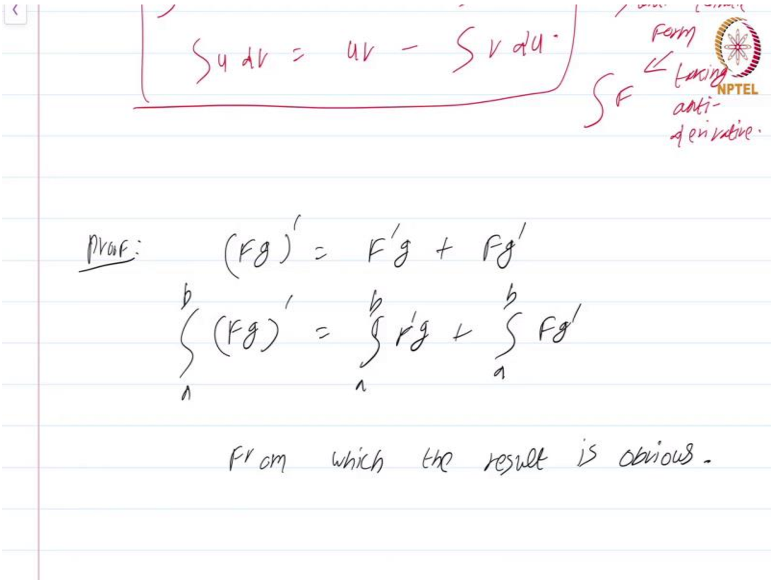

Well, how does this follow? The proof is very easy just up it just applies the famous Leibniz rule for differentiation of product. We already know that $(fg)' = f'g + fg'$ we already know this. So, $\int_a^b (fg)' = \int_a^b f'g + \int_a^b fg'$ from which it follows from which the result is obvious; the result is obvious ok.

So, two famous theorems that you have used left and right in your high school and probably in your undergraduate studies to evaluate integrals that is integration by substitution and integration by parts can both be easily justified as simple consequences of the fundamental theorem of calculus.

So, let me end with making some remarks, there will be some reference in the notes. Note that one of the consequences of the fundamental theorem of calculus is that every continuous function has an anti derivative.

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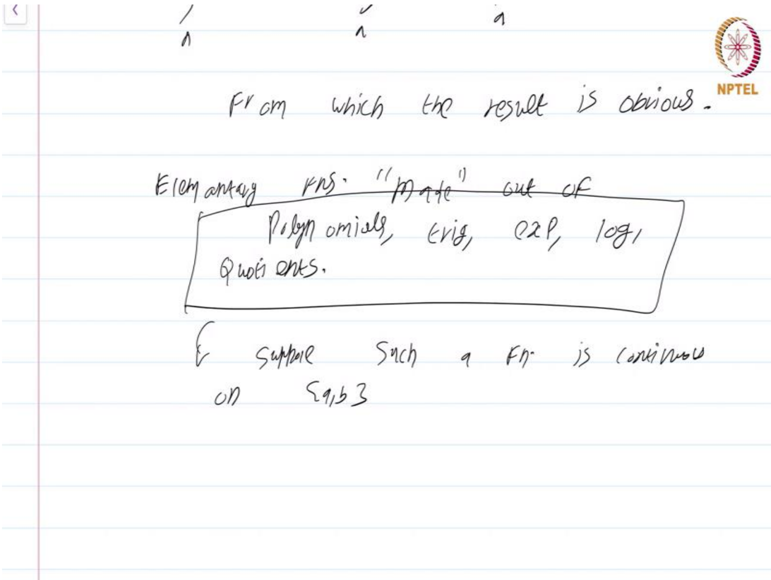

But, if you just talk about elementary functions which we are going to define very soon elementary functions these are functions made out of I will put this in quotes made out of polynomials, trigonometric functions, exponentials, logarithms, quotients.

If you just look at a function that is made out of this by made out of I just mean by adding, multiplying, subtracting, composing so on and so forth we just look at these functions if you like these are called elementary functions. I will not give a rigorous definition. I will leave it to you to check the reference in the notes.

If you just look at elementary functions, suppose such a function is continuous on the closed interval $[a, b]$, then there is no guarantee that it is anti derivative is also going to be an elementary function. We know that there is an abstract anti derivative that comes for free from the fundamental theorem of calculus, but it could get really complicated it might not be an elementary function at all.

So, do not make the mistake of thinking that just because you are given an expression that it will be easy to find what the anti derivative is. That is not the case. No doubt you are you still have nightmares of very complicated substitutions that have that seem to be pulled out of a magic hat to make the integral work. That is really needed simply because there is no algorithm to integrate in any easy manner. Please check the references if this is interesting to you.

This is a course on real analysis and you have just watched the module on the fundamental theorem of calculus.