Real Analysis - I

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Lecture – 27.3
Consequences of the Riemann-Lebesgue Theorem

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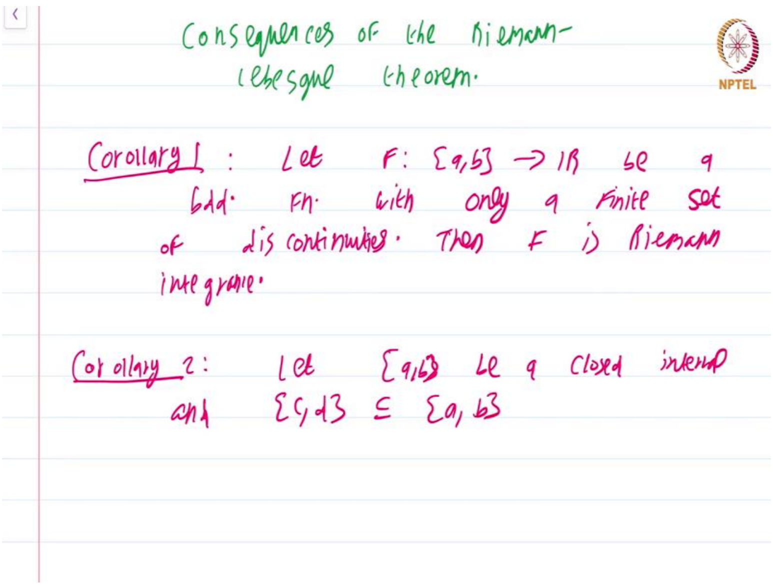

Now, that we have proved a very powerful theorem that is the Riemann Lebesgue theorem; we can get several consequences for free. I am going to start listing them one by one; most of the proofs are trivial, so what I will do is I will just indicate how to prove them and leave the details to you.

So, let me just list them as corollaries,

Corollary 1; let $f: [a, b] \longrightarrow R$ be a bounded function with only a finite set, finite set of discontinuities. Then f is Riemann integrable; this is because any finite set is of course a set of measure zero, so there is really nothing to prove.

Corollary 2; Let $[a, b]$be a closed interval be a closed interval and $[c, d] \subset [a, b]$, ok.

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Corollary 2: Let $[a,b]$ be a closed interval and $[c,d] \subseteq [a,b]$. We define

$$\chi_{[c,d]}(x) := 1 \text{ if } x \in [c,d]$$
$$= 0 \text{ otherwise}$$

indicator or characteristic fn. of $[c,d]$

Then $\chi_{[c,d]}$ is Riemann integrable.

Proof: The set of discontinuities of $\chi_{[c,d]} \subseteq \{c, d\}$.

Now, we define $\chi_{[c,d]}$; this is called the indicator or characteristic function of $[c, d]$. This can be defined more generally for all any subset; but I am just defining it for closed intervals for the time being this is by definition, $1 \; if x \in [c, d]$ and is 0 otherwise. So, this is a function whose value is 1 precisely at the points of [c, d] and 0 elsewhere, ok. Then, $\chi_{[c,d]}$ is Riemann integrable, ok. Now, the proof of this is just one line, proof; the set of discontinuities the set of discontinuities of $\chi_{[c,d]}$ is a subset of these two points c, d.

Just these two points are the only possible places, where this function can be discontinuous. So, it follows from the previous thing that says that any bounded function with only a finite set of discontinuities is automatically Riemann integrable.

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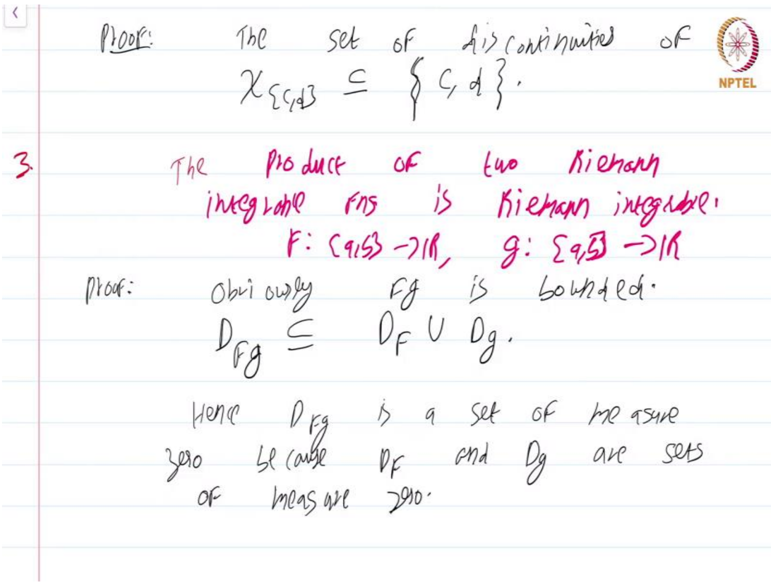

So, I am going to stop writing corollary and just write a numeral, because it saves time. The product of two Riemann integrable functions is Riemann integrable. So, let us see a proof of this; the proof is not hard. So, of course, I must write where these functions are defined $f: [a, b] \longrightarrow R$ and $g: [a, b] \longrightarrow R$, ok. Now, proof; so obviously fg is bounded.

Now, the discontinuity set D_{fg}; the set of points where the product fg is discontinuous is of course going to be a subset of the set of discontinuities of f, D_f, union the set of discontinuities of g. Note, I do not write equal to; I just write subside subset. Think why, that this will just be a subset?

Hence, D_{fg} is a set of measure zero is a set of measure zero; because D_f and D_g are sets of measure zero, sets of measure zero. This is a really short proof; let us see the next corollary.

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of measure zero.

4. $f: [a,b] \to [c,d]$ be Riemann integrable and $\phi: [c,d] \to \mathbb{R}$ be continuous then $\phi \circ f$ is Riemann integrable.

Proof: let D_f be the set of discontinuities of f then the ________ of $\phi \circ f \subseteq D_f$.
We are done because subsets of measure zero sets are measure zero.

The next corollary is somewhat involved; let $f: [a, b] \longrightarrow [c, d]$ be Riemann integrable and $\phi: [c, d] \longrightarrow R$ continuous, then $\phi \circ f$ is Riemann integrable is Riemann integrable. Again the corollary looks a bit complicated, but the proof is very easy. Proof, let D_f be the set of discontinuities of f; then the set of discontinuities $\phi \circ f$ is a subset of D_f. Why is this true?

Again, because the composition of a two functions will be continuous at a point; if f is continuous at that point and ϕ is continuous at f of that point. So, if you take a point of continuity of f; then $\phi \circ f$ will automatically be continuous at that point, therefore the only point of discontinuities of $\phi \circ f$ will be a subset of D_f.

Again, think why I write subset and not equal to, ok. And we are done because subsets of measure zero, set are measure zero for the same reason as the previous are measure zero.

Now, I had rambled on a bit about why I want to prove the somewhat technical Riemann Lebesgue theorem saying that, many consequences will become fairly straightforward and I have demonstrated at least a few of them. To make this demonstration even more potent, I want you to sit down on a nice afternoon and try to prove 3 and 4 directly from the condition that a function is Riemann integrable if and only if for each epsilon greater than zero; we can find a partition P such that $U(f, P) - L(f, P) < \varepsilon$.

Directly from that, can you solve problems 3 and 4? And you will realize that the techniques that you use to prove 3 and 4, put together with a little bit of tweak can be used to prove the Riemann Lebesgue theorem itself. So, why not just prove the Riemann Lebesgue theorem and get it done for once and for all, ok. So, that is one observation that I wanted to make.

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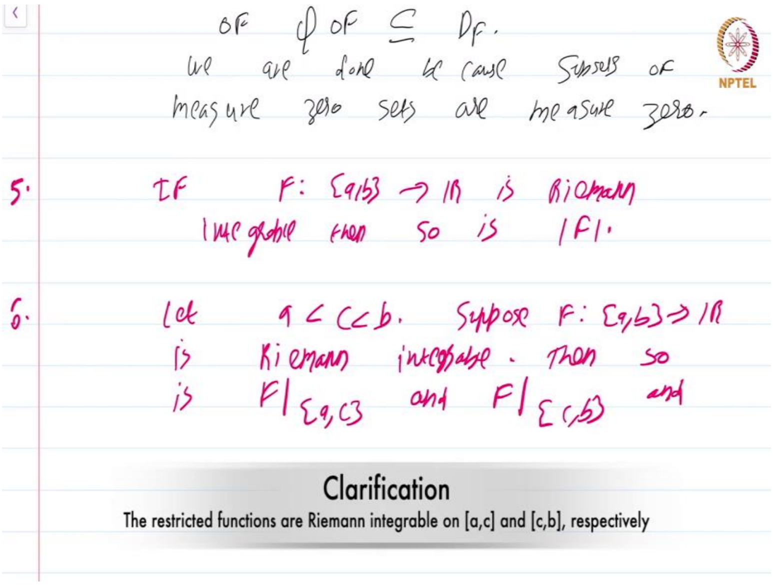

Let us move on with the corollaries number 5; if $f: [a, b] \longrightarrow R$ is Riemann integrable, then so is $|f|$. This is obvious, because the modulus is a continuous function and you can treat modulus of f as absolute value composed with the function f and apply the previous corollary. So, it follows immediately. 6, let a<c< b; suppose $f: [a, b] \longrightarrow R$ is Riemann integrable is Riemann integrable, ok.

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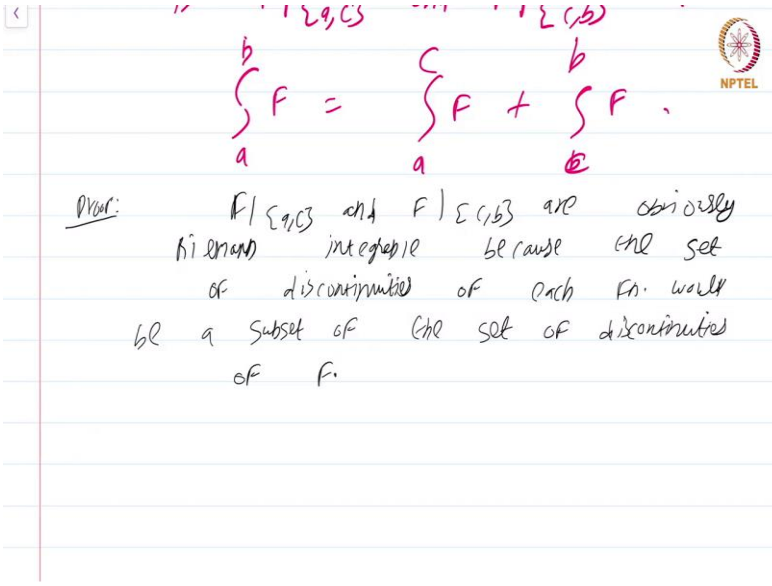

Then so is f restricted to $[a, c]$ and f restricted to [c, b] and $\int_a^b f = \int_a^c f + \int_c^b f$. Again the proof of this is very easy;

Proof: f restricted to $[a, c]$ and f restricted to $[c, b]$ are obviously Riemann integrable,.

Because the set of discontinuities of each function would be a subset of the set of discontinuities of f, that is obvious, ok. So, both functions will obviously be Riemann integrable.

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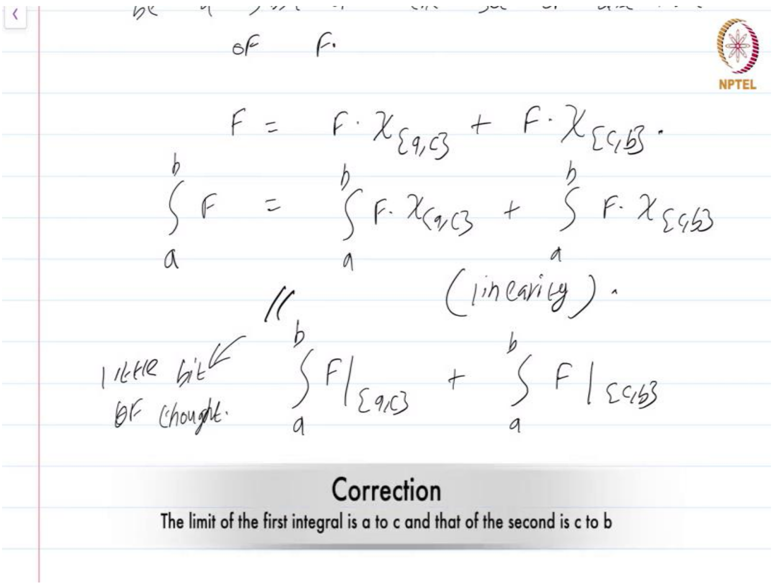

Now, observe that I can write f as $f \cdot \chi_{[a,c]} + f \cdot \chi_{[c,b]}$. I can write the function f as a product of the function $f = f\ \chi_{[a,c]} + f \cdot \chi_{[c,b]}$. Now, with a little bit of thought, it is now clear that $\int_a^b f$ is nothing but $\int_a^b f \cdot \chi_{[a,c]} + \int_a^b f \cdot \chi_{[c,b]}$. This just follows from linearity, which we have established in an earlier module, this is just linearity, ok.

Now, the little bit of thought part comes now, this is nothing but $\int_a^b f\,|_{[a,c]} + \int_a^b f\,|_{[c,b]}$. So, this is the part that requires little bit of thought. When I say little bit, I really mean little bit of thought, ok.

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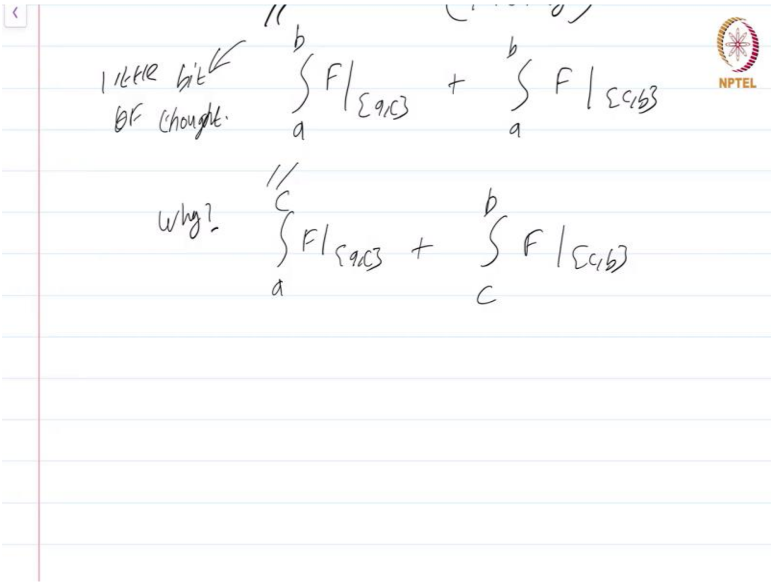

Now, the original proposition says, it is $\int_a^c f + \int_c^b f$, that is rather easy to see; $\int_a^b f\,|_{[a,c]} + \int_a^b f\,|_{[c,b]}$ is actually same as $\int_a^c f\,|_{[a,c]} + \int_c^b f\,|_{[c,b]}$. Why? Please check why this is true; again this also requires just a little bit of thought.

So, we have now seen several consequences of the Riemann Lebesgue theorem; none of them are particularly difficult, except the last one which requires some thought, but they are all straightforward and easy consequences of the Riemann Lebesgue theorem.

I hope you come out of this module feeling that the effort that was involved in understanding the Riemann Lebesgue theorem is certainly worth, worth the hard work. Hope that you solve whatever bits that I have left for you.

This is a course on Real Analysis, and you have just watched the module on Consequences of the Riemann Lebesgue Theorem.