Complex Analysis

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Lecture No – 3.1

Power Series

In the last week, we defined the notion of complex differentiability. We saw that complex differentiable functions also satisfy the laws of calculus, namely linearity, the product rule, quotient rule and also the chain rule. Thereafter, we saw a few examples of complex differentiable functions and in particular we noted that polynomials in the variable z are complex differentiable in the entire complex plane; they are entire functions.

This week we begin by discussing the notion of power series. Power series is an infinite degree variant of a polynomial inside its disk of convergence. A power series behaves very similar to polynomials both analytically and algebraically. Let us start this lecture by defining a power series around a point z_0 in the complex plane.

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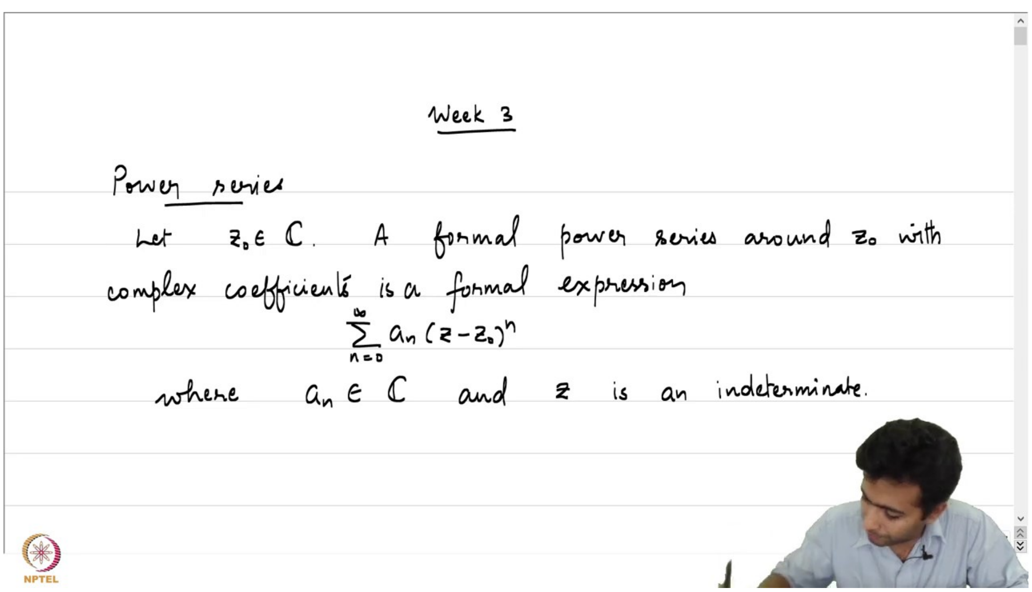

Let $z_0 \in \mathbb{C}$. A formal power series around z_0 with complex coefficients is a formal expansion,

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n \in \mathbb{C}$ and z is an indeterminate.

We could ask whether a formal power series converges at a given point $z \in \mathbb{C}$. For example, at z_0, the formal power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely.

Another example is,consider the geometric series, $\sum_{n=0}^{\infty} z^n$, then for any z with $|z| \geq 1$, the summands, $|z^n| > 1$ and does not converge to 0. Hence $\sum_{n=0}^{\infty} z^n$ does not converge.

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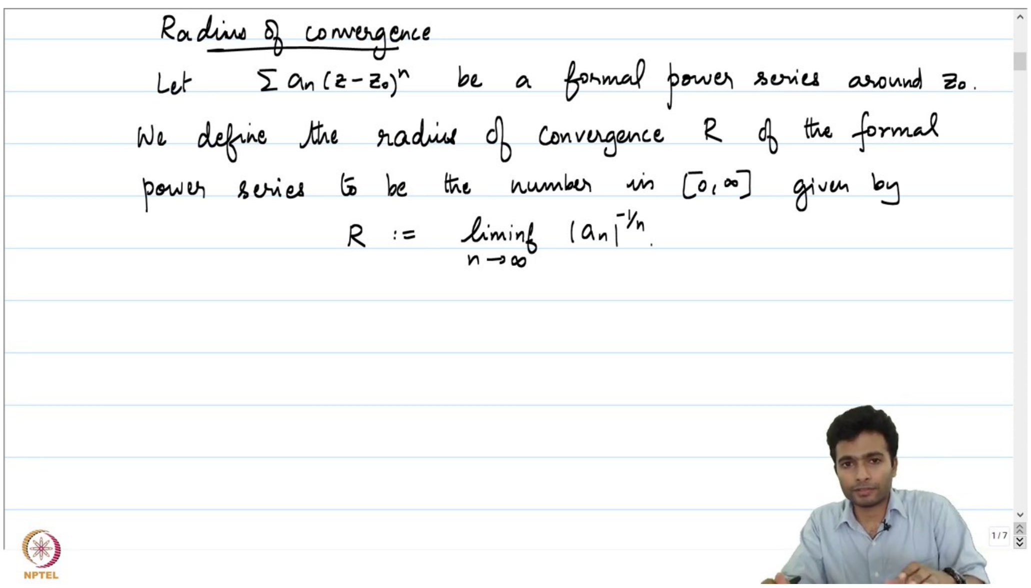

Radius of Convergence

Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a formal power series around z_0. We define the radius of convergence R of the formal power series to be the number in $[0, \infty]$ given by

$$R := \liminf_{n \to 0} |a_n|^{-1/n}$$

.

The radius of convergence, as the name suggests, is a quantity which tells us something about the convergence of the given formal power series which is captured in the next proposition.

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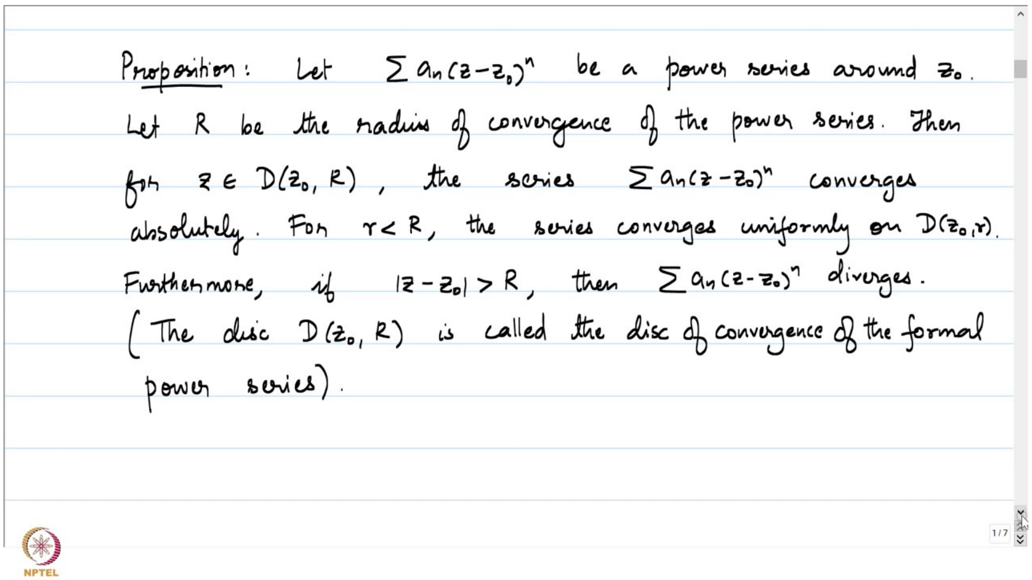

PROPOSITION 1. *Let* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *be a power series around* z_0*. Let* R *be the radius of convergence of the power series. Then for* $z \in D(z_0, R)$*, the series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *converges absolutely. For* $r < R$*, the series converges uniformly on* $D(z_0, r)$*. Furthermore, if* $|z-z_0| > R$*, then* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *diverges.*

(The disc $D(z_0, R)$ is called the disc of convergence of the formal power series).

PROOF. Let $z \in \mathbb{C}$ such that $|z-z_0| > R$. Then $\exists$ infinitely many $n \in \mathbb{N}$ such that

$$|a_n|^{-1/n} < |z-z_0| \implies |a_n|^{-1/n}|z-z_0| > 1 \implies |a_n(z-z_0)^n| > 1$$

for infinitely many $n \in \mathbb{N} \to (*)$. Since the summands does not converge to 0(by $(*)$), we can conclude that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ does not converge.

Let $z \in D(z_0, R) \implies |z-z_0| < r < R$ for some $r > 0$. Since $R = \liminf_{n\to\infty} |a_n|^{-1/n}$, $\exists\, N \in \mathbb{N}$ such that $\forall n > N, |a_n|^{-1/n} > r \implies |a_n| r^n < 1 \forall n > N$.

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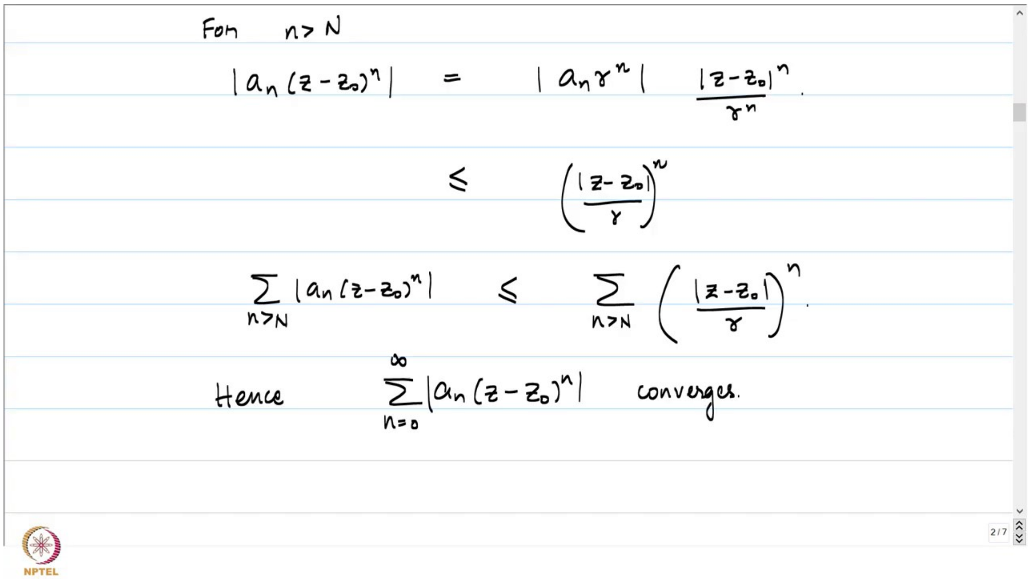

For $n > N$,

$$|a_n(z-z_0)^n| = |a_n r^n|\frac{|z-z_0|^n}{r^n}$$
$$\le \left(\frac{|z-z_0|}{r}\right)^n$$
$$\sum_{n>N} a_n(z-z_0)^n \le \sum_{n>N}\left(\frac{|z-z_0|}{r}\right)^n.$$

Hence $\sum_{n=0}^{\infty} |a_n(z-z_0)^n|$ converges. That is $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely.
Let $r < R$ and R_1 be such that $r < R_1 < R$. $\exists N \in \mathbb{N}$ such that $\forall n > N, |a_n|^{-1/n} > R_1 > r$. For $n > 1$ and $z \in D(z_0, r)$,

$$|a_n(z-z_0)^n| = |a_n R_1^n|\frac{|z-z_0|^n}{R_1^n} \le \left(\frac{r}{R_1}\right)^n$$

. Since $\sum_{n=0}^{\infty}\left(\frac{r}{R_1}\right)^n$ is a convergent series, given $\varepsilon > 0, \exists N_0 > N$ such that $\sum_{n\ge N_0}\left(\frac{r}{R_1}\right)^n < \varepsilon$.

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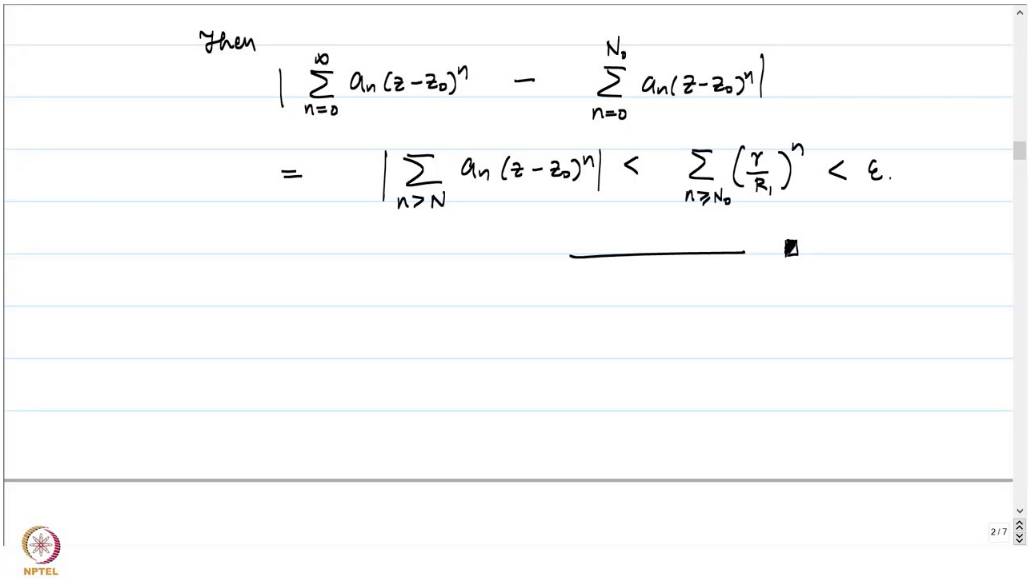

Then,

$$\left|\sum_{n=0}^{\infty} a_n(z-z_0)^n - \sum_{n=0}^{N_0} a_n(z-z_0)^n\right| = \left|\sum_{n>N_0} a_n(z-z_0)^n\right|$$
$$< \sum_{n\geq N_0}\left(\frac{r}{R_1}\right)^n$$
$$< \varepsilon.$$

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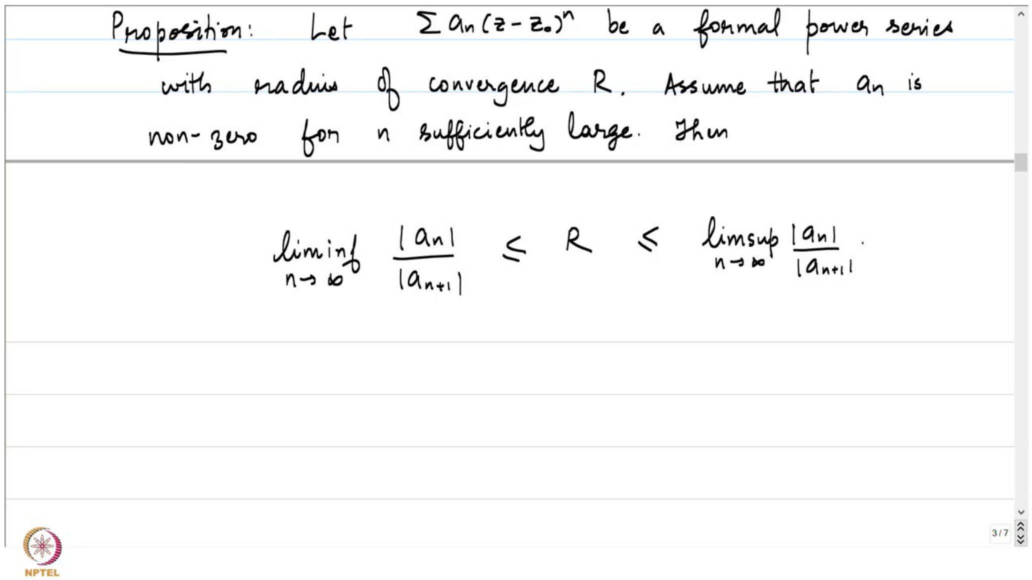

PROPOSITION 2. *Let* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *be a formal power series with radius of convergence* R*. Assume that* a_n *is non-zero for* n *sufficiently large. Then*

$$\liminf_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \leq R \leq \limsup_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}.$$

PROOF. Let $R_1 = \liminf_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$. Let $r < R_1$. Then $\exists N \in \mathbb{N}$ such that $\forall n > N$,

$$\frac{|a_n|}{|a_{n+1}|} > r \implies |a_{n+1}|r < |a_n|.$$

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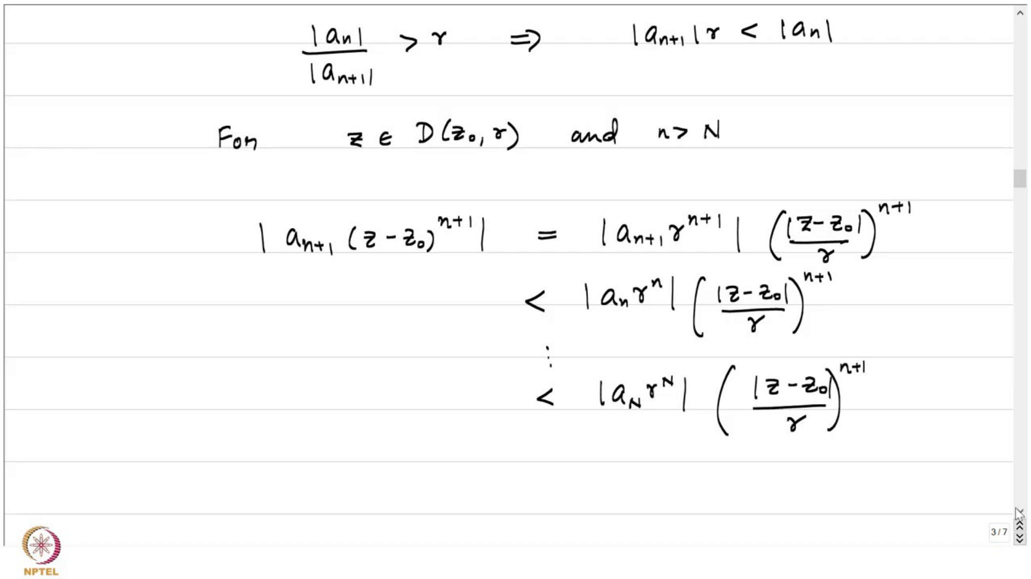

For $z \in D(z_0, r)$ and $n > N$,

$$
\begin{aligned}
|a_{n+1}(z-z_0)^{n+1}| &= |a_{n+1}| \left(\frac{|z-z_0|}{r}\right)^{n+1} \\
&< |a_n r^n| \left(\frac{|z-z_0|}{r}\right)^{n+1} \\
&\vdots \\
&< \left|a_N r^N\right| \left(\frac{|z-z_0|}{r}\right)^{n+1}.
\end{aligned}
$$

Hence $\sum_{n \geq N} |a_n (z-z_0)^n| \leq \left|a_N r^N\right| \sum_{n \geq N} \left(\frac{|z-z_0|}{r}\right)^n$.

Hence $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges $\implies D(z_0, r) \subseteq \{z : |z-z_0| \leq R\} \implies r \leq R$. Since this is true for all $r < R_1$, we have $R_1 \leq R \to (*)$.

Let $R_2 = \limsup_{n \to \infty} \frac{|a_n|}{|a_{n+1}|}$. Let $r > R_2$ and let $z \in \mathbb{C}$ be such that $|z - z_0| > r$. Since $r > R_2, \exists N \in \mathbb{N}$ such that $\forall n > N$,

$$
\frac{|a_n|}{|a_{n+1}|} < r \implies |a_{n+1} r| > |a_n|.
$$

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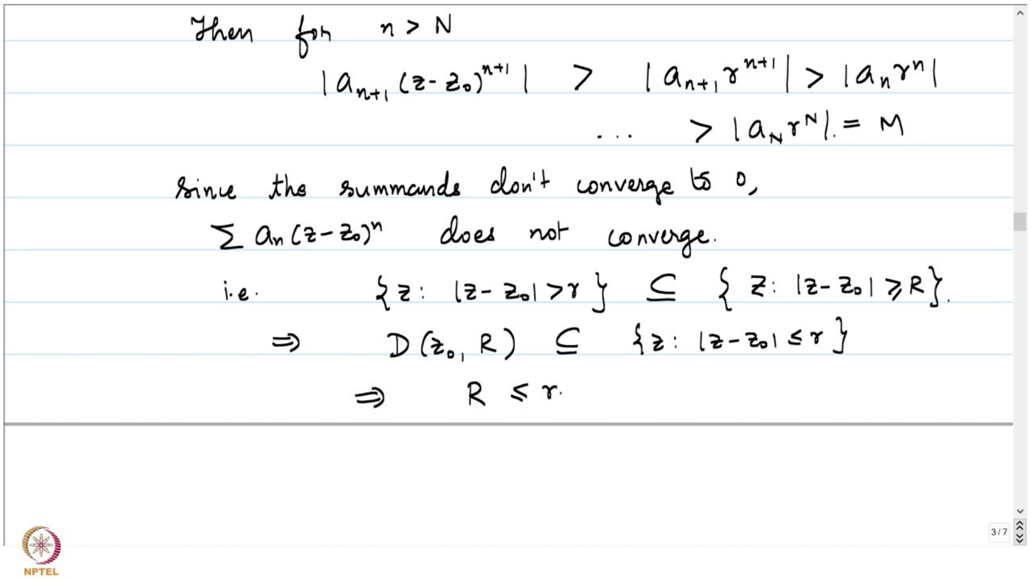

Then for $n > N$,

$$|a_{n+1}(z-z_0)^{n+1}| > |a_{n+1}r^{n+1}| > |a_n r^n| \cdots > |a_N r^N| = M(\text{say})$$

Since the summands do not converge to 0, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ does not converge.

That is, $\{z : |z-z_0| > r\} \subseteq \{z : |z-z_0| \geq R\} \implies D(z_0, R) \subseteq \{z : |z-z_0| \leq r\} \implies R \leq r$.

Since this is true for all $r > R_2 \implies R \leq R_2 \to (**)$.

By $(*)$ and $(**)$, $\liminf_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \leq R \leq \limsup_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$. $\square$

EXAMPLE 3.

- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

  Here $a_n = \frac{1}{n!}$. Now, $\frac{|a_n|}{|a_{n+1}|} = \frac{(n+1)!}{n!} = (n+1)$. Hence the radius of convergence is infinite.

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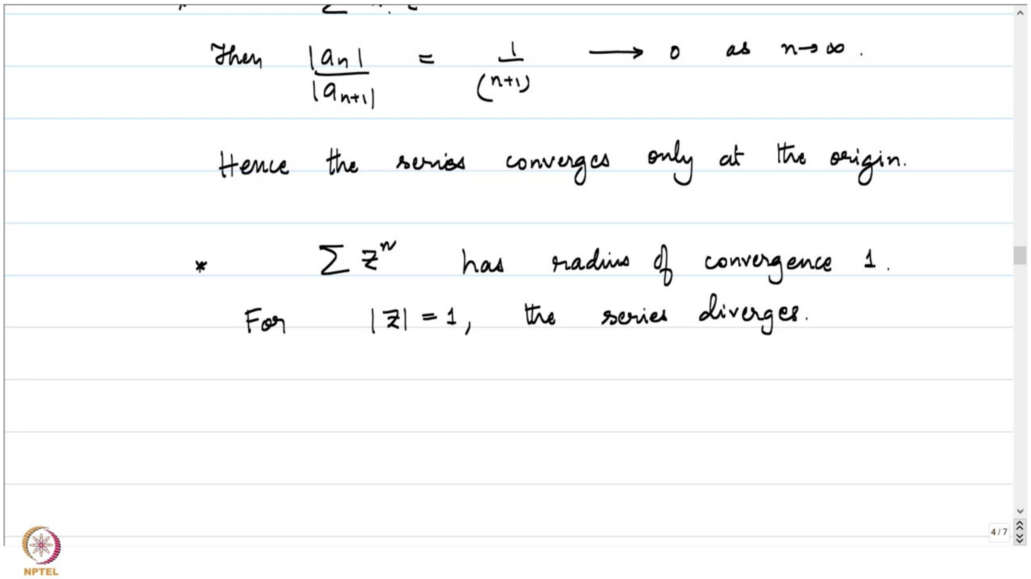

- $\sum_{n=0}^{\infty} n!z^n$.

  Then $\frac{|a_n|}{|a_{n+1}|} = \frac{(n!}{(n+1)!} = \frac{1}{(n+1)} \to 0$ as $n \to \infty$. Hence the series converges only at the origin.
- $\sum_{n=0}^{\infty} z^n$ has radius of convergence 1. For $|z| = 1$, the series diverges. For $z \in D(0,1)$, we have

$$
\begin{aligned}
z\sum_{n=0}^{\infty} z^n &= \sum_{n=0}^{\infty} z^{n+1} \\
&= \sum_{n=1}^{\infty} z^n \\
&= \sum_{n=0}^{\infty} z^n - 1.
\end{aligned}
$$

On $D(0,1)$, we have $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

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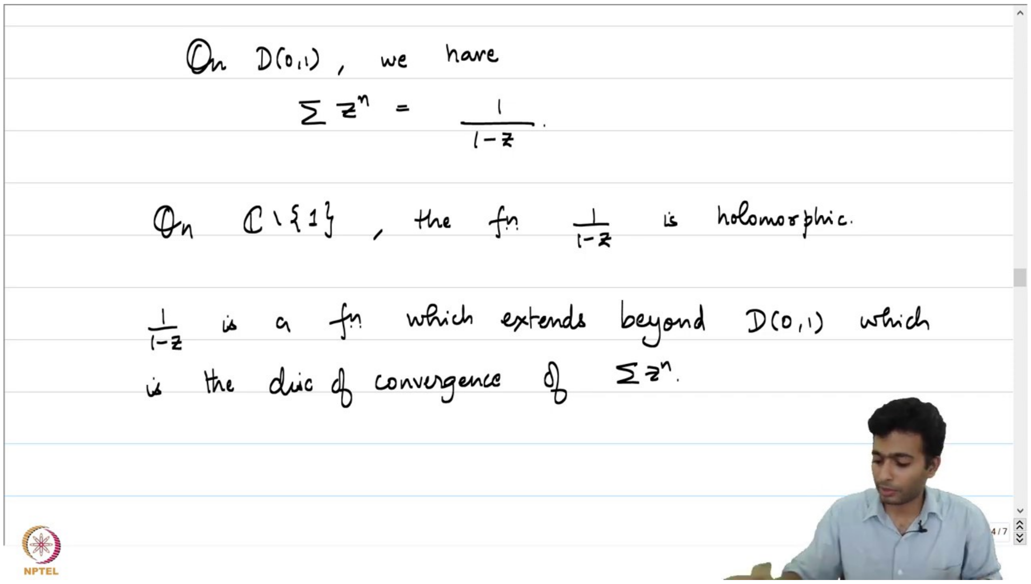

On $\mathbb{C} \setminus \{1\}$, the function $\dfrac{1}{1-z}$ is holomorphic. Now note that $\dfrac{1}{1-z}$ is a function which extends beyond $D(0,1)$ which is the disc of convergence of $\sum_{n=0}^{\infty} z^n$.

- $\sum_{n=0}^{\infty} \dfrac{z^n}{n^2}$ has a radius of convergence 1.

  For $z \in \mathbb{C}$ such that $|z| = 1$, $\sum_{n=0}^{\infty} \dfrac{z^n}{n^2}$ converges.

Abel's Theorem: *Let $F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series with a positive radius of convergence R and suppose $z_1 = z_0 + Re^{i\theta}$ be a point such that $F(z_1)$ converges. Then* $\lim_{r \to R^-} F(z_0 + re^{i\theta}) = F(z_1)$.

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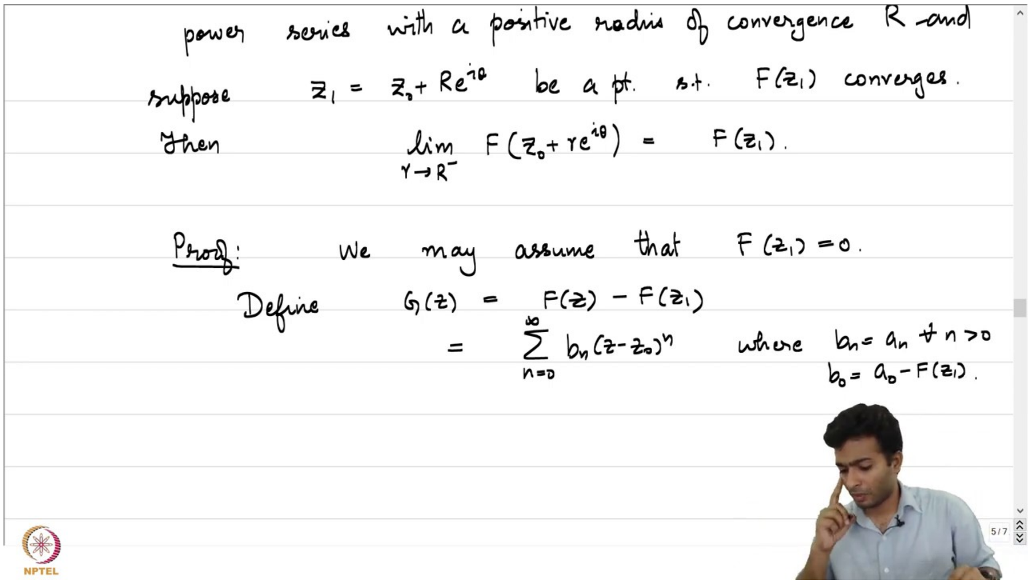

PROOF. We may assume that $F(z_1) = 0$.

Define $G(z) = F(z) - F(z_1) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$ where $b_n = a_n \forall n > 0, b_0 = a_0 - F(z_1)$. If $\lim_{r \to R^-} G(z_0 + re^{i\theta}) = 0$, then $\lim_{r \to R^-} (F(z_0 + re^{i\theta}) - F(z_1)) = 0$. Hence we shall assume that $F(z_1) = 0$.

We can also assume $z_0 = 0$.

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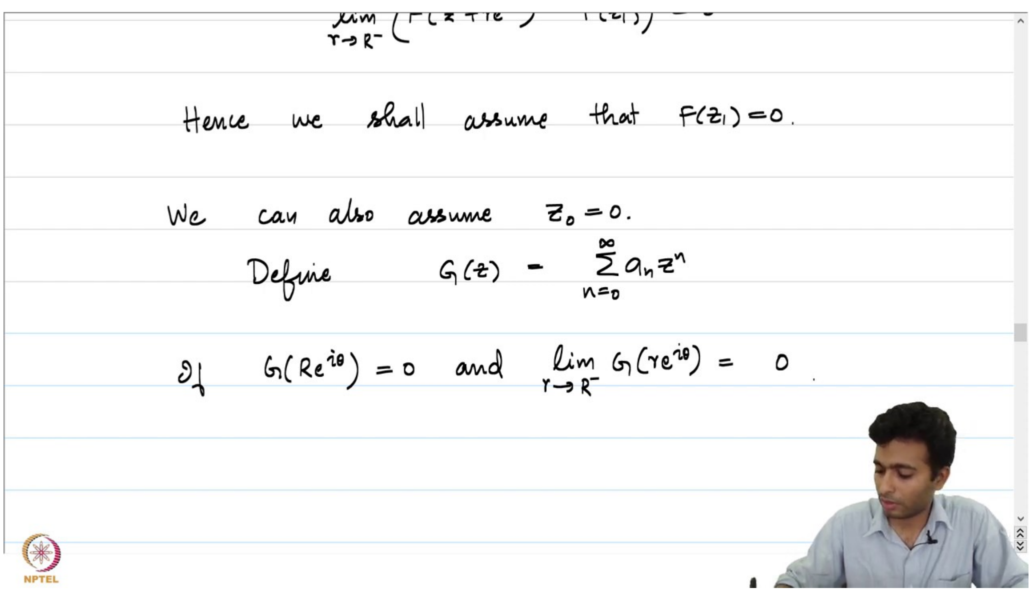

Define $G(z) = \sum_{n=0}^{\infty} a_n z^n$. If $G(re^{i\theta}) = 0$ and $\lim_{r \to R^-} G(re^{i\theta}) = 0$, then reader should verify that $\lim_{r \to R^-} G(re^{i\theta}) = \lim_{r \to R^-} F(z_0 + re^{i\theta}) = 0$. Hence we shall assume that $z_0 = 0$.
Thus if we prove for the power series around 0, we can translate it can conclude that it is true for all power series around any arbitrary point.

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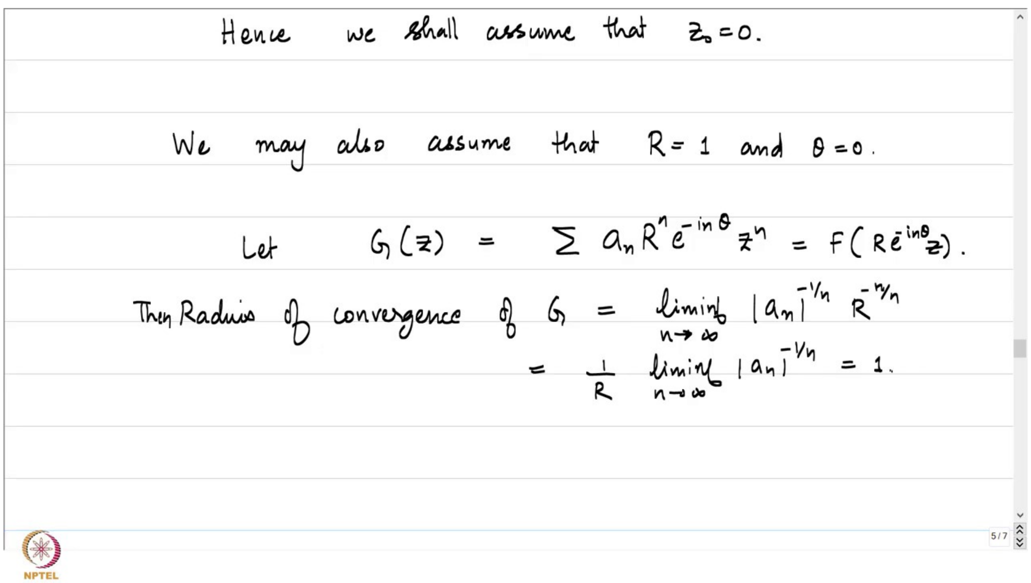

We may also assume that $R = 1$ and $\theta = 0$.
Let $G(z) = \sum a_n R^n e^{-in\theta} z^n = F(Re^{-in\theta} z)$.
The radius of convergence of $G = \liminf_{n\to\infty} |a_n|^{-1/n} R^{-n/n} = \frac{1}{R} \liminf_{n\to\infty} |a_n|^{-1/n} = 1$. If $G(1) = 0$ and $\lim_{r \to 1^-} G(r) = 0$, then $\lim_{r \to R^-} F(re^{i\theta}) = 0$.
Hence to prove the theorem, it is enough to prove the following:

Let $\sum a_n$ be a series converging to 0. Then $\lim_{r \to 1^-} \sum a_n r^n = 0$.

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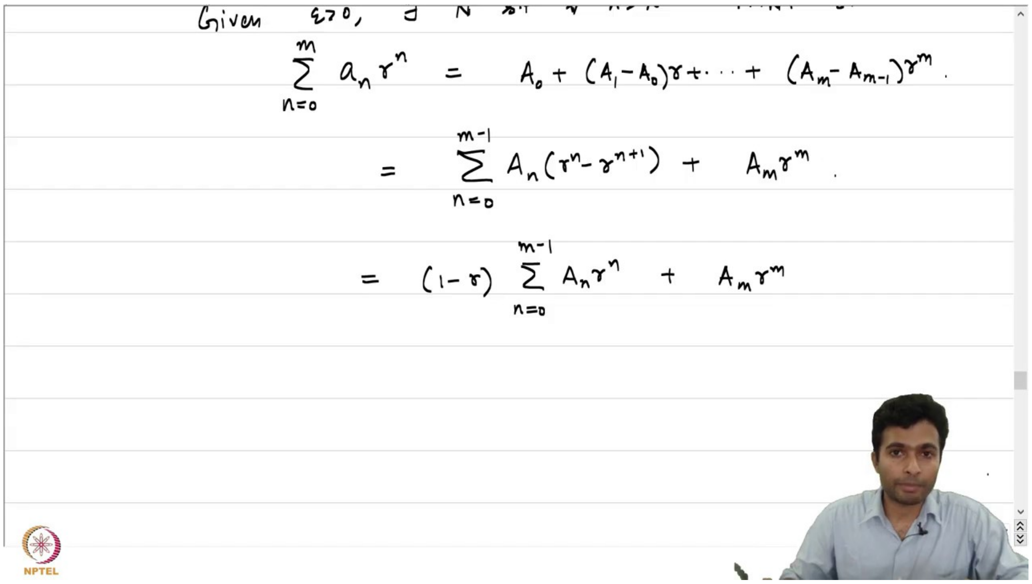

Let $A_n = a_0 + a_1 + \cdots + a_n$. Given $\varepsilon > 0, \exists N$ such that $\forall n > N, |A_n| < \frac{\varepsilon}{2}$.

$$\sum_{n=0}^{m} a_n r^n = A_0 + (A_1 - A_0)r + \cdots + (A_m - A_{m-1})r^m$$
$$= \sum_{n=0}^{m-1} A_n(r^n - r^{n+1}) + A_m r^m$$
$$= (1-r)\sum_{n=0}^{m-1} A_n r^n + A_m r^m.$$

Hence $\sum_{n=0}^{\infty} a_n r^n = (1-r)\sum_{n=0}^{m-1} A_n r^n \to (*)$.
For $r < 1$ and given $\varepsilon > 0$ as above, R.H.S of $(*)$ becomes,

$$\left|(1-r)\sum_{n=0}^{N} A_n r^n + (1-r)\sum_{n=N}^{\infty} A_n r^n\right| \le (1-r)\left|\sum_{n=0}^{N} A_n r^n\right| + (1-r)\sum_{n=N}^{\infty} |A_n| r^n$$
$$< (1-r)\left|\sum_{n=0}^{N} A_n r^n\right| + (1-r)\frac{\varepsilon}{2}\frac{r^N}{(1-r)}$$
$$< (1-r)\left|\sum_{n=0}^{N} A_n r^n\right| + \frac{\varepsilon}{2}.$$

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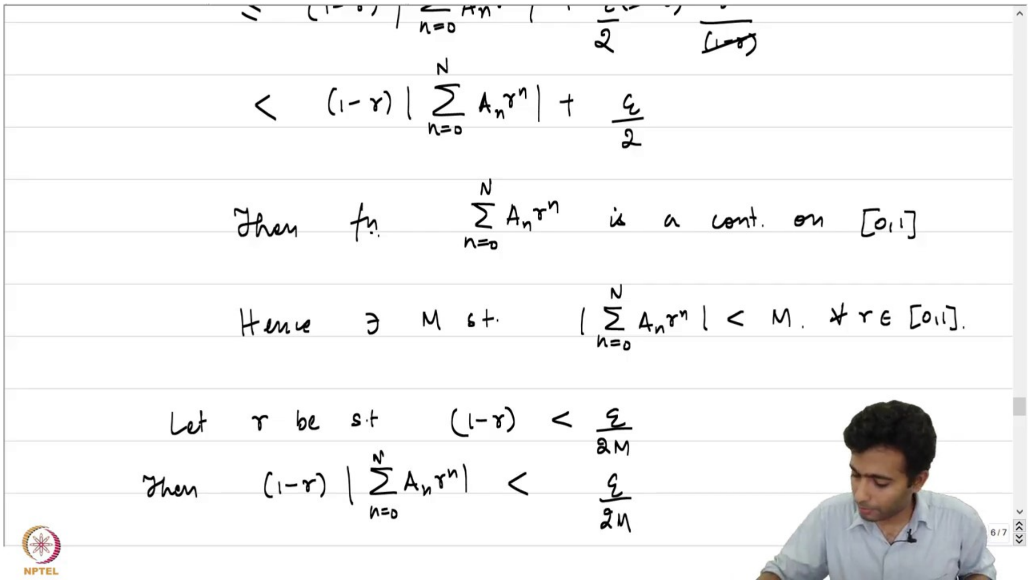

Then the function $\sum_{n=0}^{N} A_n r^n$ is a continuous on $[0,1]$. Hence $\exists M$ such that $\left| \sum_{n=0}^{N} A_n r^n \right| < M \; \forall \; r \in [0,1]$.

Let r be such that $(1-r) < \dfrac{\varepsilon}{2M}$. Then $(1-r) \left| \sum_{n=0}^{\infty} A_n r^n \right| < \dfrac{\varepsilon}{2M} M = \dfrac{\varepsilon}{2}$.

Hence $\lim_{r \to 1^-} \sum a_n r^n = \lim_{r \to 1^-} \left((1-r) \sum_{n=0}^{\infty} A_n r^n \right) = 0$.

$\square$