Complex Analysis

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Lecture No – 2.4

Complex differentiability

At the very heart of the study of Real analysis, lies the notion of differentiability and integration and these two notions are tied together by the very beautiful fundamental theorem of calculus. In Complex analysis also, the things are quite similar. At the very heart of the study of Complex analysis, the notion of complex differentiability and complex line integrals lie and they are also tied together by a variant of the fundamental theorem of calculus.

There is a further variant of this fundamental theorem called the Cauchy's theorem, which wends certain amount of rigidity to complex differentiable functions thereby making the theory very beautiful. In this lecture we will define complex differentiability and explore some of its properties.

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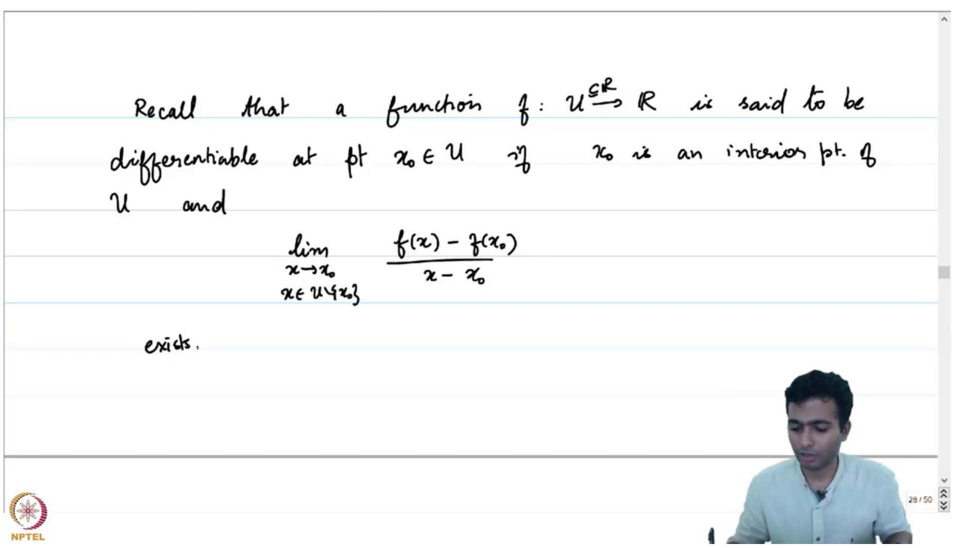

Recall that a function $f : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be differetiable at a point $x_0 \in U$ if x_0 is an interior point of U and $\lim\limits_{\substack{x \to x_0 \\ x \in U \setminus \{x_0\}}} \frac{f(x)-f(x_0)}{x-x_0}$ exists. The limit is denoted by $f'(x_0)$ which is called the derivative of f at x_0.

Complex differentiability

Let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$. We say that f is complex differentiable at a point $z_0 \in \Omega$ if z_0 is an interior point and $\lim\limits_{\substack{z \to z_0 \\ z \in \Omega \setminus \{z_0\}}} \frac{f(z)-f(z_0)}{z-z_0}$ exists. The limit is deoted by $f'(z_0)$ or $\frac{df}{dz}(z_0)$. If f is complex differentiable at every point $z \in \Omega$, then we say that f is complex differentiable on Ω or holomorphic on Ω.

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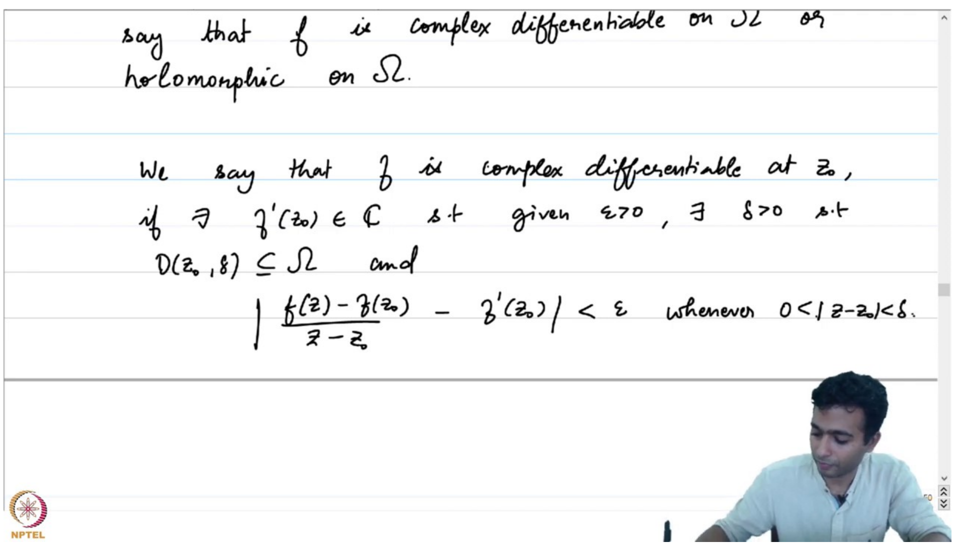

There also many other variants for the definition of complex differentiability. Another important variant is $\epsilon - \delta$ definition for complex differentiability:
We say that f is complex differentaible at z_0, if $\exists\ f'(z_0) \in \mathbb{C}$ such that, given $\epsilon > 0, \exists \delta > 0$ such that $D(z_0, \delta) \subseteq \Omega$ and $\left| \frac{f(z)-f(z_0}{z-z_0} - f'(z_0) \right| < \epsilon$ whenever $0 < |z - z_0 < \delta$.

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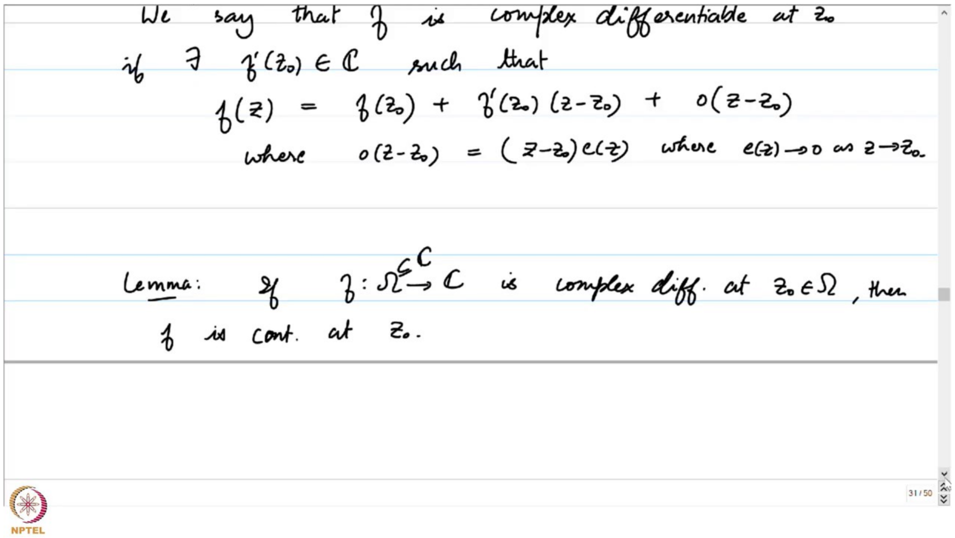

Another definition is in terms of linear approximation,

We say that f is complex differentiable at z_0 if $\exists\, f'(z_0) \in \mathbb{C}$ such that

$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0)$, where $o(z - z_0) = (z - z_0)e(z), e(z) \to 0$ as $z \to z_0$.

LEMMA 1. *Let $\Omega \subseteq \mathbb{C}$. If $f : \Omega \longrightarrow \mathbb{C}$ is complex differentiable at z_0, then f is continuous at z_0.*

The proof the lemma is immediate from the definition of complex differentiability.

EXERCISE 2. Let $f(z) = z^n$ be defined on $\mathbb{C}$. Fix $z_0 \in \mathbb{C}$. Then for $z \neq z_0$,

$$\begin{aligned}
\frac{f(z) - f(z_0)}{z - z_0} &= \frac{z^n - z_0^n}{z - z_0} \\
&= z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1} \\
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= z_0^{n-1} + z_0^{n-1} \cdots + z_0^{n-1} \qquad (n - \text{terms}) \\
&= n z_0^{n-1}
\end{aligned}$$

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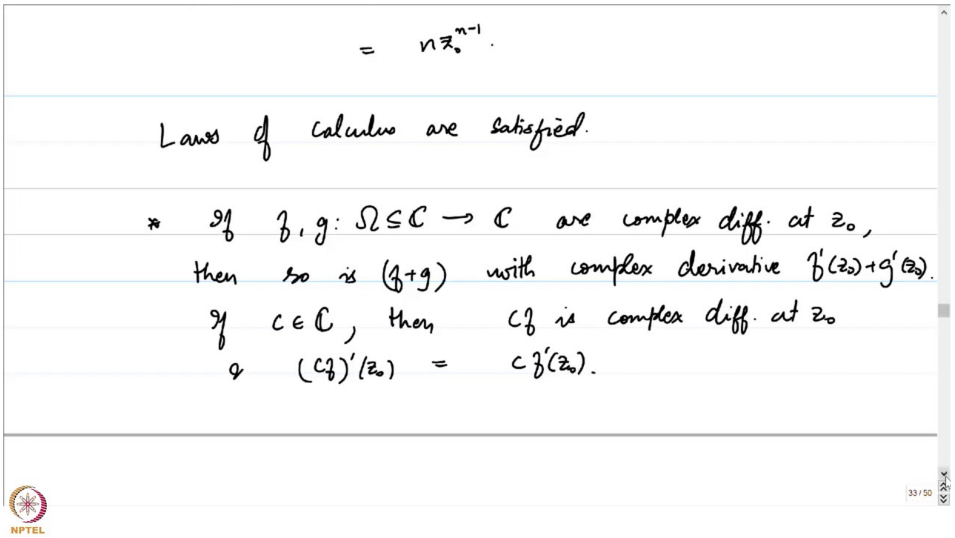

Laws of Calculus are satisfied:

- Linearity

  If $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ are complex differentiable at $z_0 \in \Omega$, then so is $f+g$ with complex derivative $f'(z_0)+g'(z_0)$. If $c \in \mathbb{C}$, then cf is complex differentiable at z_0 and $(cf)'(z_0) = cf'(z_0)$.

- Product rule

  If $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be complex differentiable at $z_0 \in \Omega$. Then (fg) is complex differentiable at z_0 with derivative $f'(z_0)g(z_0)+f(z_0)g'(z_0)$.

- Quotient rule

  If $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be complex differentiable at $z_0 \in \Omega$ and suppose that g does not vanish at z_0. Then by the continuity, g does not vanish in a neighborhood of D of z_0. Then $\dfrac{f}{g}$ is complex differentiable at z_0.

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{(g(z_0))^2}.$$

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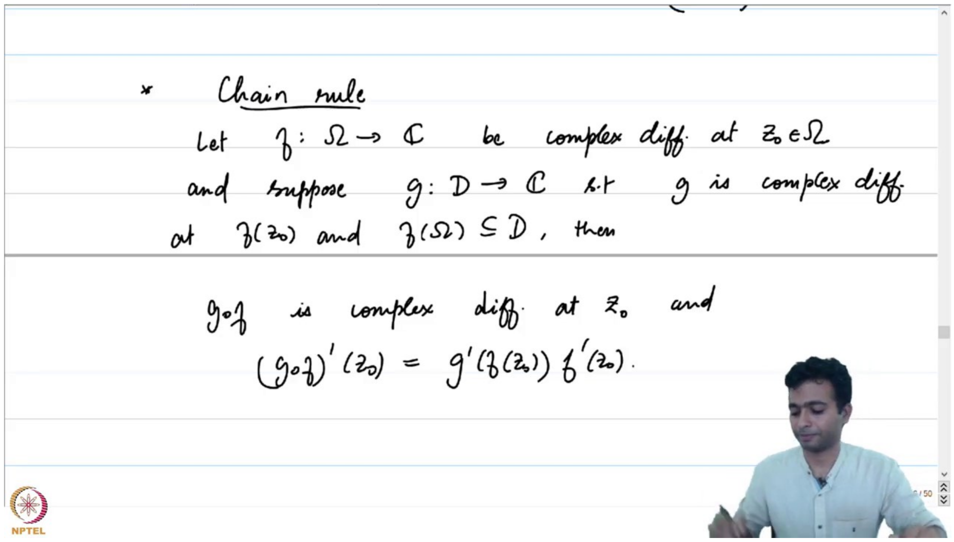

- <u>Chain rule</u> Let $f : \Omega \longrightarrow \mathbb{C}$ be complex differentiable at $z_0 \in \Omega$ and suppose $g : D \longrightarrow \mathbb{C}$ such that g is complex differentiable at $f(z_0)$ and $f(\Omega) \subseteq D$, then $g \circ f$ is complex differentiable at z_0 and $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$.

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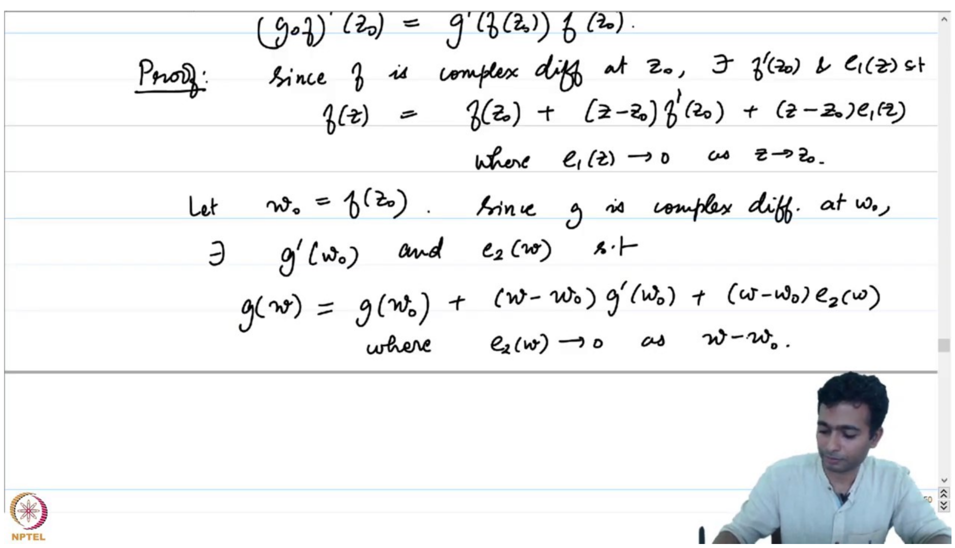

PROOF. Since f is complex differentiable at $z_0, \exists f'(z_0)$ and $e_1(z)$ such that $f(z) = f(z_0) + (z - z_0)f'(z_0) + (z - z_0)e_1(z)$, where $e_1(z) \to 0$ as $z \to z_0$.

Let $w_0 = f(z_0)$. Since g is complex differentiable at $w_0, \exists\, g'(w_0)$ and $e_2(w)$ such that, $g(w) = g(w_0) + (w - w_0)g'(w_0) + (w - w_0)e_2(w)$, where $e_2(w) \to 0$ as $w \to w_0$.

Since f is complex differentiable at z_0, it is continuous at z_0. Hence $f(z) \to w_0$ whenever $z \to z_0$.

$$\begin{aligned} g(f(z)) &= g(f(z_0)) + \big(f(z) - f(z_0)\big)\, g'(f(z_0)) + (f(z) - f(z_0))e_2(f(z)) \\ &= \big(f(z) - f(z_0)\big)\big(g'(f(z_0)) + e_2(f(z))\big) \\ &= (z - z_0)\big(f'(z_0) + e_1(z)\big)\big(g'(f(z_0)) + e_2(f(z))\big) \end{aligned}$$

$$\begin{aligned} \lim_{\substack{z \to z_0 \\ z \in \Omega \setminus \{z_0\}}} \frac{g(f(z)) - g(f(z_0))}{z - z_0} &= \lim_{\substack{z \to z_0 \\ z \in \Omega \setminus \{z_0\}}} \big(f'(z_0) + e_1(z)\big)\big(g'(f(z_0)) + e_2(f(z))\big) \\ &= f'(z_0)g'(f(z_0)) \end{aligned}$$

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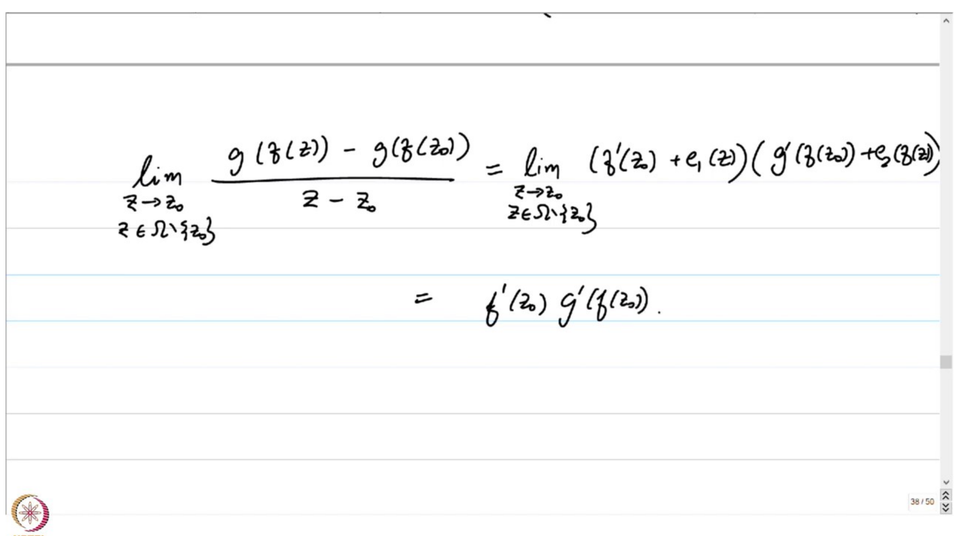

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EXERCISE 3. Prove the linearity, product rule and quotient rule for complex derivative.

Functions which are complex differentiable on $\mathbb{C}$ are called **entire functions**. For example, $z \to z^n$ is an entire function.
By using the laws of Calculus,we have, $p(z) = a_0 + a_1 z + \cdots + a_d z^d$ is an entire function. Moreover, we know that, for $z_0 \in \mathbb{C}$, $p'(z_0) = a_1 + 2a_2 z_0 + \cdots + d a_d z_0^{d-1}$.

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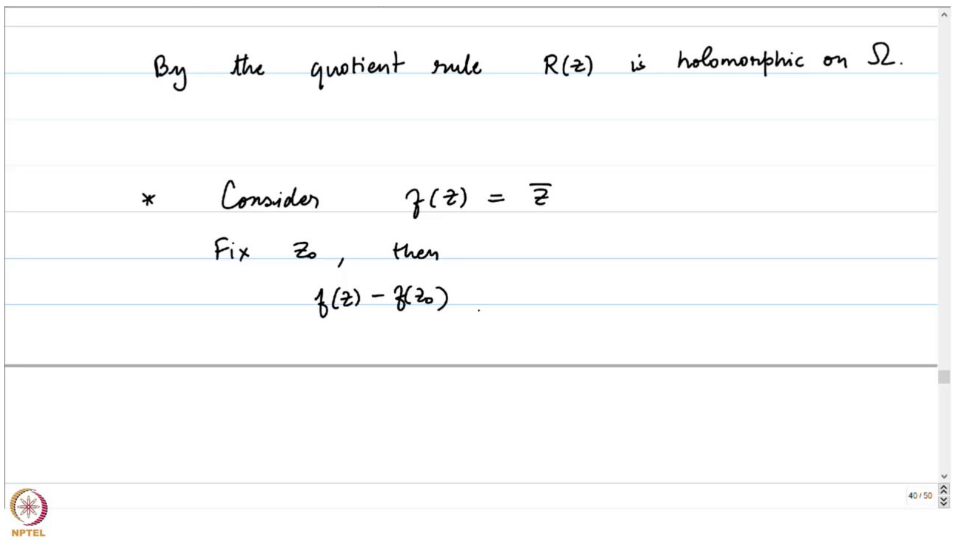

Let $R(z) = \dfrac{p(z)}{q(z)}$ on Ω such that $q(z) \neq 0 \forall z \in \Omega$. By the quotient rule $R(z)$ is holomorphic on Ω.

- Consider $f(z) = \overline{z}$. Fix $z_0 \in \mathbb{C}$, then $\dfrac{f(z) - f(z_0)}{z - z_0} = \dfrac{\overline{z} - \overline{z_0}}{z - z_0}$. Let $z = z_0 + h, \overline{z} = \overline{z_0} + \overline{h}$, where $h \neq 0$. $\lim\limits_{z \to z_0} = \dfrac{f(z) - f(z_0)}{z - z_0} = \lim\limits_{h \to 0} \dfrac{\overline{h}}{h}$. Suppose $h \to 0$ along the real axis, then $\lim\limits_{h \to 0} \dfrac{\overline{h}}{h} = 1$. Suppose $h \to 0$ along the imaginary axis, then $\lim\limits_{h \to} \dfrac{\overline{h}}{h} = -1$. Hence the limit does not exists $\implies f(z)$ is not holomorphic.

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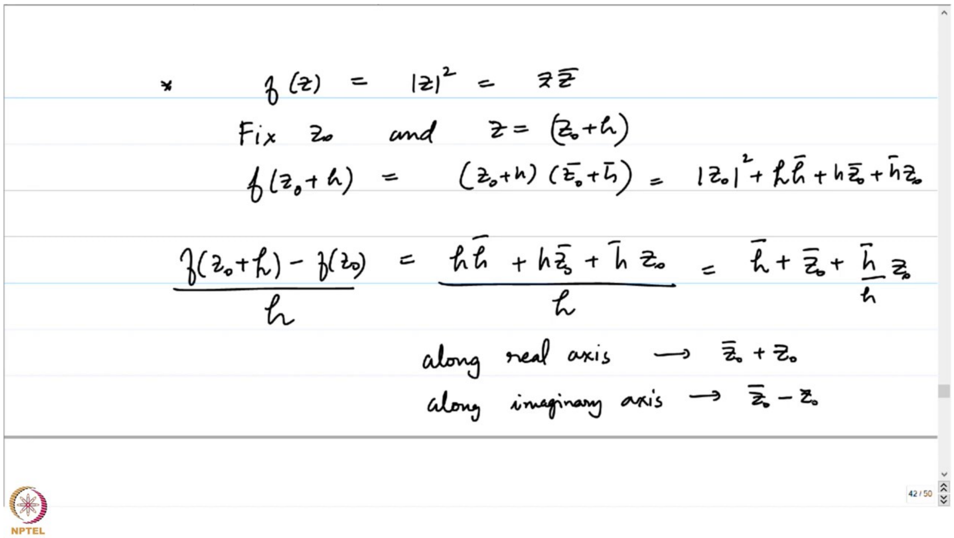

- Let $f(z) = |z|^2 = z\overline{z}$. Fix $z_0 \in \mathbb{C}$ and $z = (z_0 + h)$.Then,

$$f(z_0 + h) = (z_0 + h)(\overline{z_0} + \overline{h}) = |z_0|^2 + |h|^2 + h\overline{z_0} + \overline{h}z_0 \implies$$

$$\begin{aligned} \frac{f(z_0 + h) - f(z_0)}{h} &= \frac{h\overline{h} + h\overline{z_0} + \overline{h}z_0}{h} \\ &= \overline{h} + \overline{z_0} + \frac{\overline{h}}{h} z_0 \end{aligned}$$

Now on taking the limit $h \to 0$ along real axis will obtain $\overline{z_0} + z_0$ and along the imaginary axis will obtain $\overline{z_0} - z_0$. Since the limits are not equal, the limit does not exists. Hence f is not holomorphic.