Complex Analysis

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Lecture No – 2.3

Functions on the complex plane

In the last lecture we discussed isometrics on the complex plane; isometries were distance preserving complex valued functions which were defined on the entire complex plane. Throughout this course we will be interested in studying complex valued functions which are defined on open connected subsets of the complex plane.

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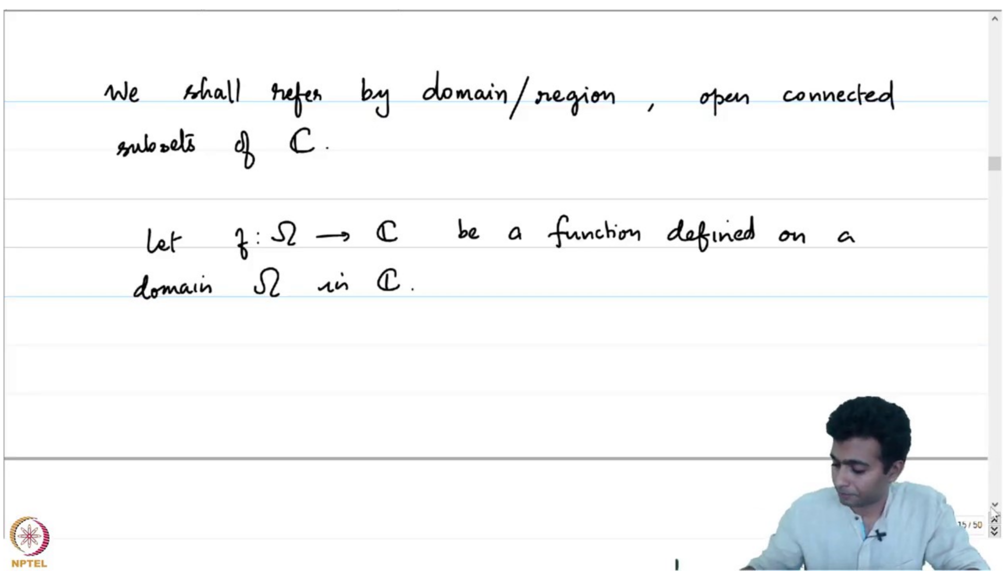

We shall refer to open connected subsets of $\mathbb{C}$ by the terms, domain or region.

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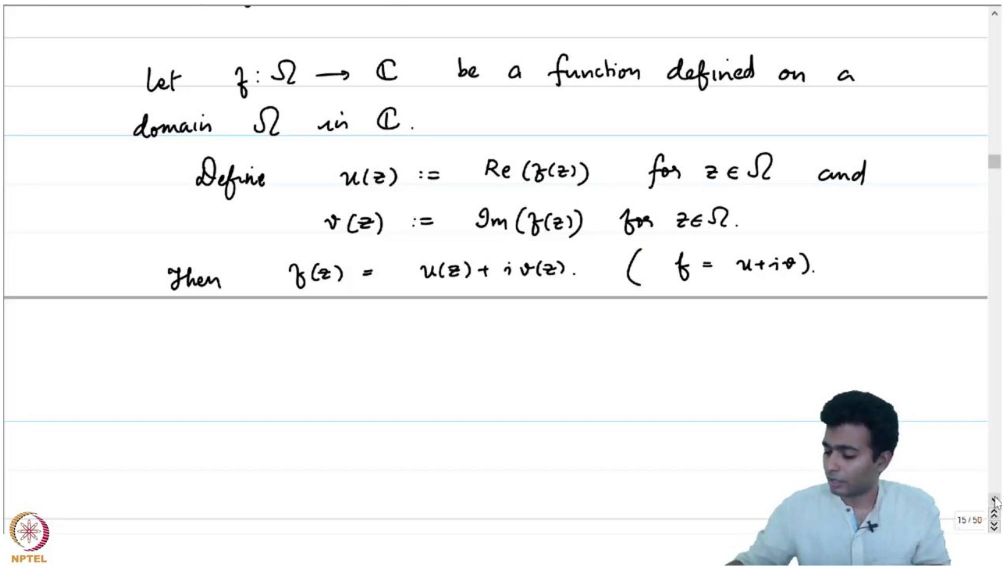

Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f : \Omega \longrightarrow \mathbb{C}$ be a complex valued function. Define $u(z) := \mathfrak{Re}(f(z))$ and $v(z) := \mathfrak{Im}(f(z))$ for $z \in \Omega$. Then $f(z) = u(z) + i v(z)$.

EXAMPLE 1. Consider $p : \mathbb{C} \longrightarrow \mathbb{C}$ given by $p(z) = a_0 + a_1 z + \cdots + a_d z^d$. Since $\mathbb{C}$ can be identified with $\mathbb{R}^2$, we have $z = x + iy$ and thus $p(z)$ can be identified as $p(x, y)$,then $p(z) = p(x, y) = p_1(x, y) + i p_2(x, y)$. But notice that, every polynomial in two variable may not be written as a polynomial in z itself. If $p(x, y) = x$, then $p(z) = \frac{z+\overline{z}}{2}$. Here p is a polynomial which involves both z and $\overline{z}$.

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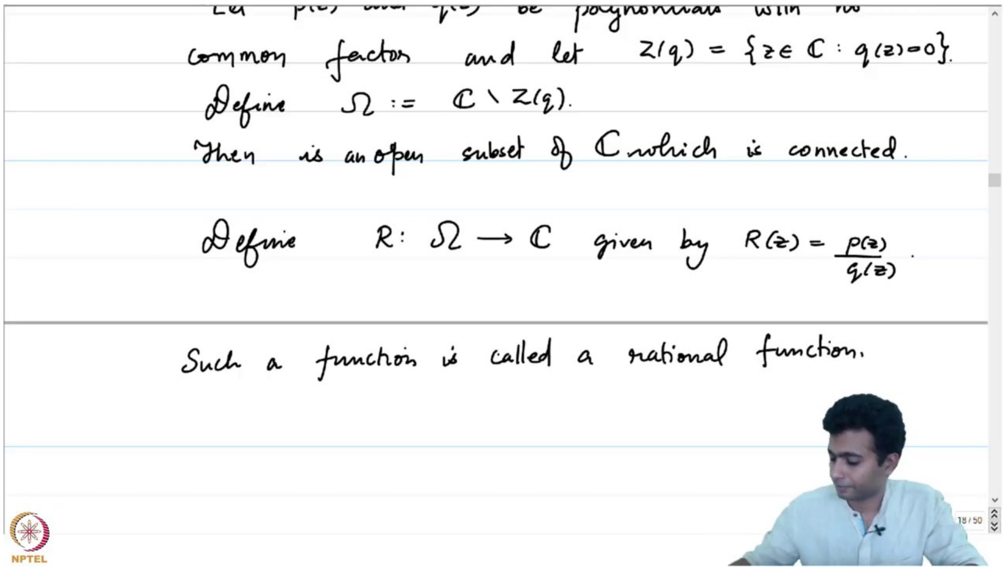

EXAMPLE 2 (**Rational functions**). Let $p(z)$ and $q(z)$ be polynomials with no common factors and let $Z(q) = \{z \in \mathbb{C} : q(z) = 0\}$. Note that $q(z)$ is a polynomial it has a degree, say d, then the number of points where q vanishes (counting multiplicities) will be less than or equal to d. Then $Z(q)$ is finite and hence closed. Define $\Omega := \mathbb{C} \setminus Z(q)$. Then, since $Z(q)$ is finite and closed, Ω is an open connected subset of $\mathbb{C}$.

Define $R : \Omega \longrightarrow \mathbb{C}$ given by $R(z) := \frac{p(z)}{q(z)}$. Then such a function is called rational function.

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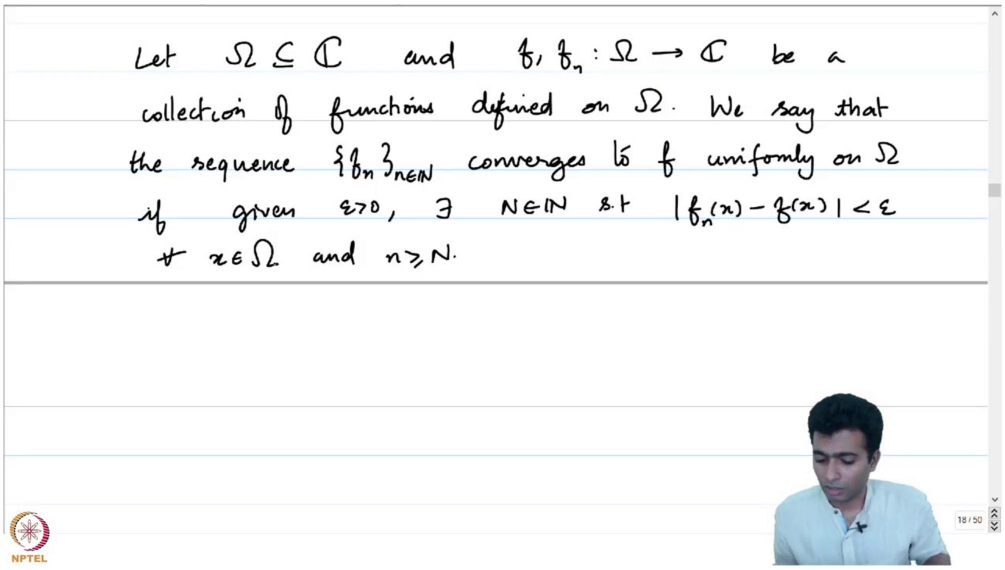

DEFINITION 1 (**Uniform convergence**). Let $\Omega \subseteq \mathbb{C}$ and $f, f_n :\longrightarrow \mathbb{C}$ be a collection of functions defined on Ω. We say that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on Ω if given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ $\forall x \in \Omega$ and $n \geq N$.

EXERCISE 3. Let $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions on $\Omega \subseteq \mathbb{C}$ which converges uniformly to a function f on Ω. Then f is a continuous function on Ω.

DEFINITION 2 (**Absolute Convergence**). We say that a series $\sum\limits_{n=1}^{\infty} z_n$ converges absolutely if $\sum\limits_{n=1}^{\infty} |z_n|$ converges.

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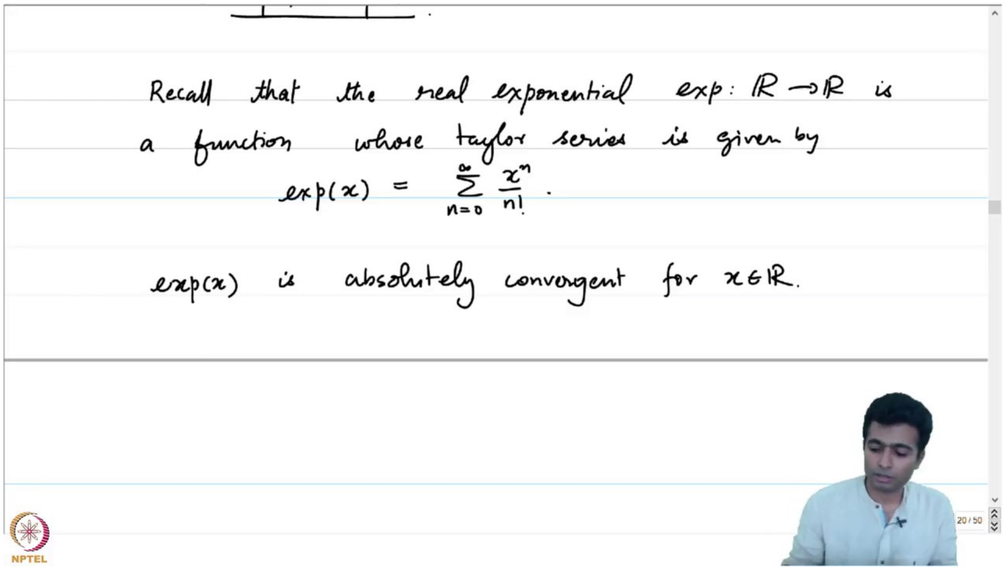

Complex exponential

We are familiar with the real exponential function $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ whose Taylor series is given by,

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

$\exp(x)$ is absolutely convergent for $x \in \mathbb{R}$.

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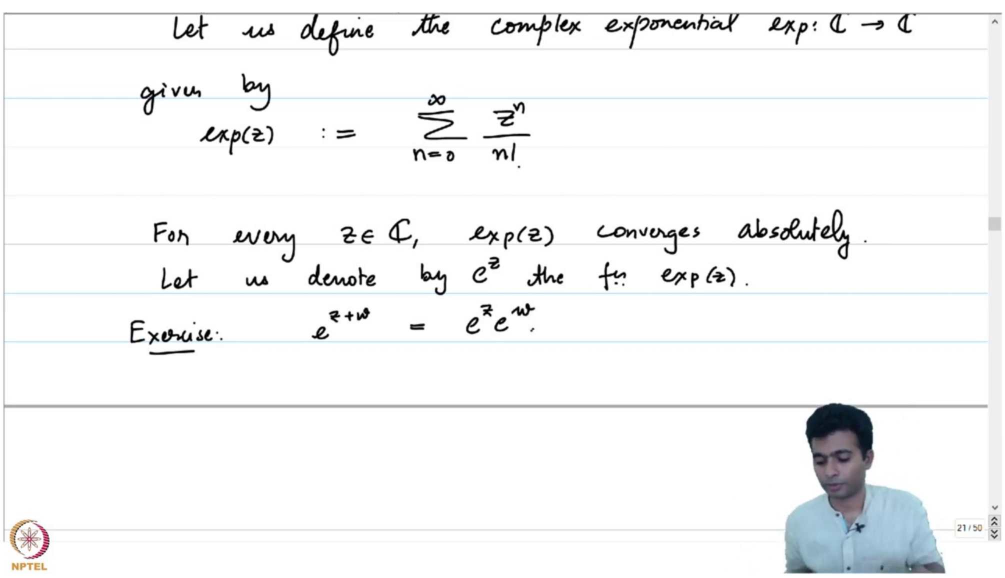

Let us define the define the complex exponential $\exp : \mathbb{C} \longrightarrow \mathbb{C}$ given by

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

For every $z \in \mathbb{C}$, $\exp(z)$ converges absolutely. Let us denote the function $\exp(z)$ by e^z.

EXERCISE 4. Prove that, for every $z, w \in \mathbb{C}$, $e^{z+w} = e^z \cdot e^w$.

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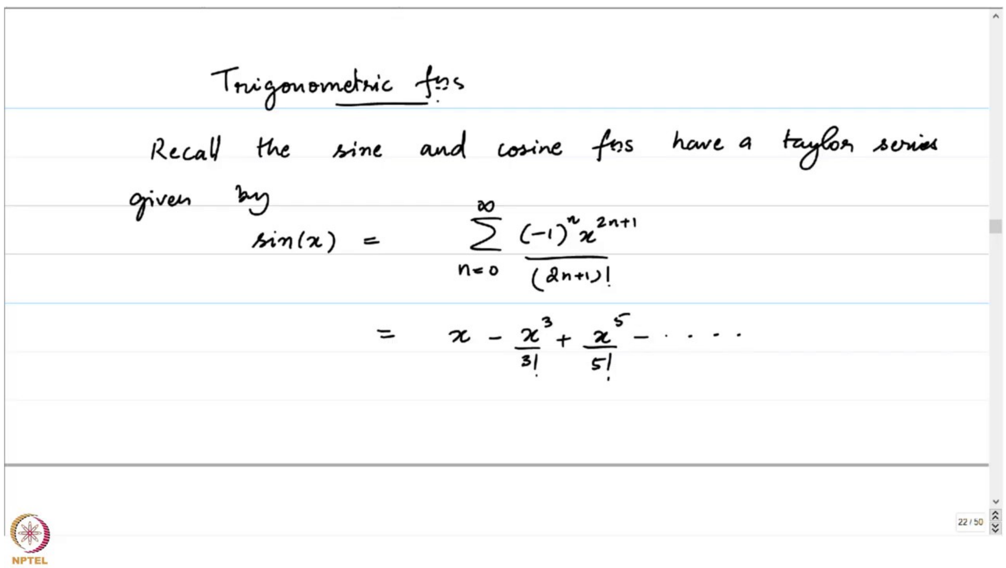

Trigonometric functions We know that in the real variable cases, sine and cosine functions have a Taylor series given by

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots
\end{aligned}$$

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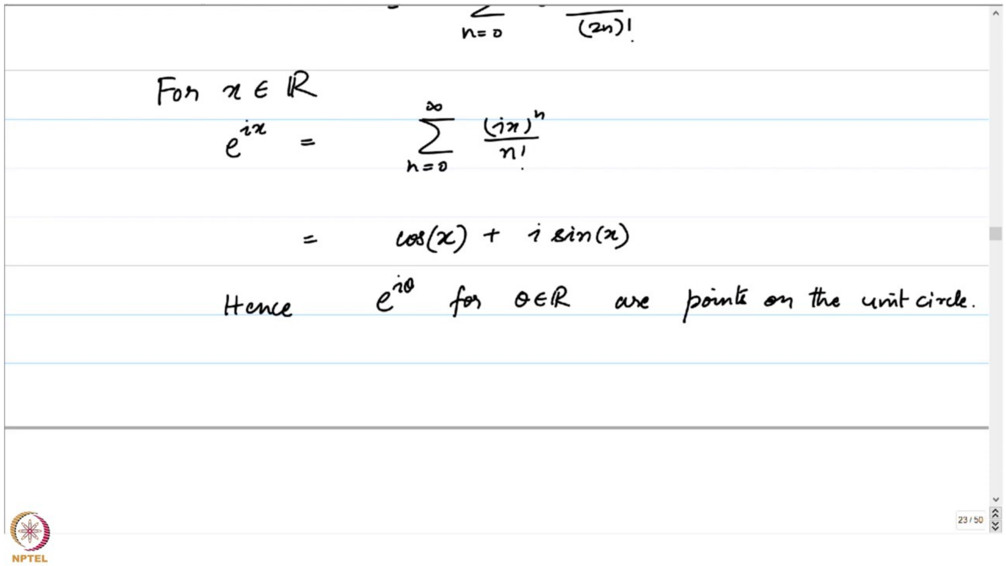

For $x \in \mathbb{R}$, $e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \cos x + i \sin x$. Hence, for $\theta \in \mathbb{R}$, $e^{i\theta}$ are points the unit circle.

EXERCISE 5.

(1) Find the values of
   - $e^{2\pi i}$
   - $e^{\pi i}$
   - $e^{i\frac{\pi}{2}}$

(2) Prove that $e^{i\theta} \cdot e^{i\phi} = e^{i(\theta+\phi)}$ and $\overline{e^{i\theta}} = e^{-i\theta}$.

Notice that $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$.

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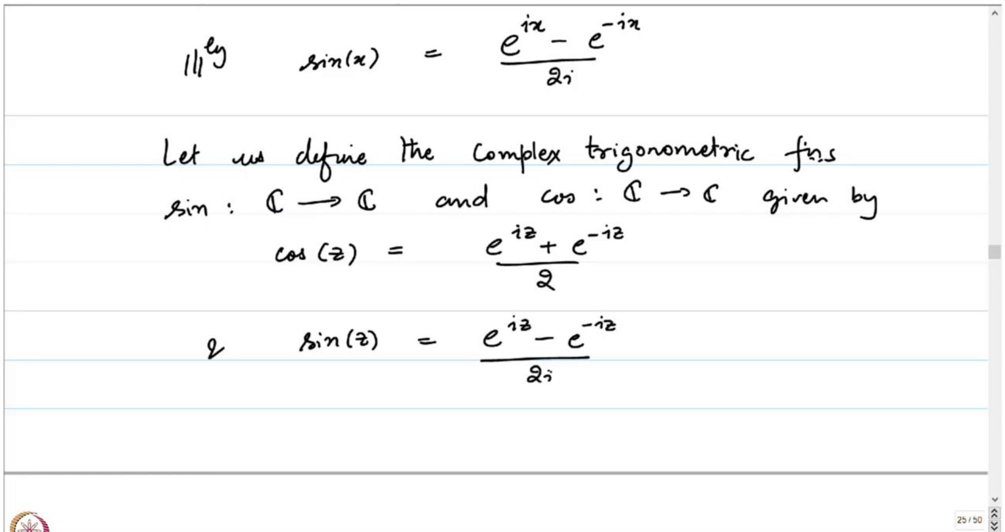

Let us define the complex trigonometric functions $\sin : \mathbb{C} \longrightarrow \mathbb{C}$ and $\cos : \mathbb{C} \longrightarrow \mathbb{C}$ given by,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

EXERCISE 6. Prove that $\cos^2 z + \sin^2 z = 1$.

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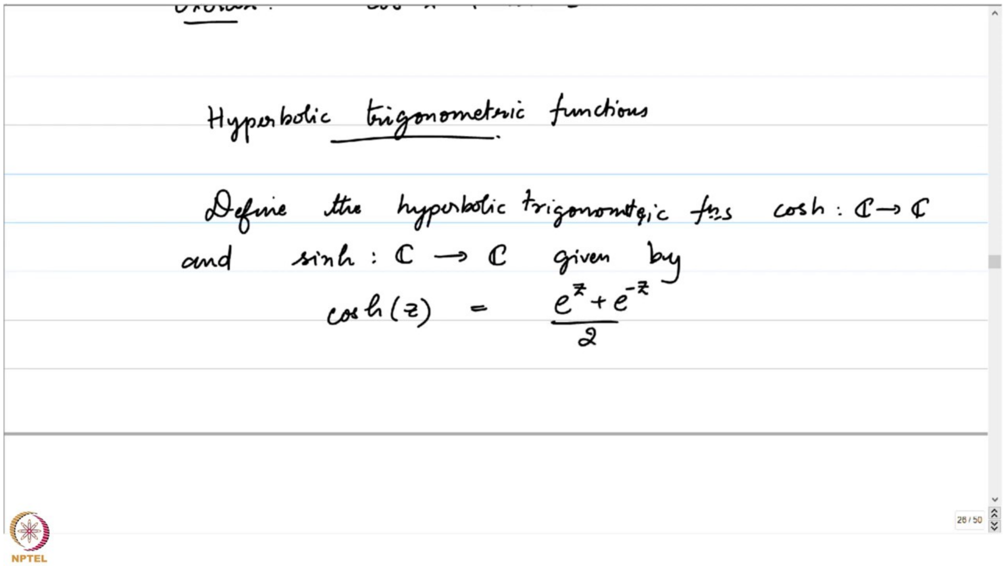

Hyperbolic trigonometric functions

Define hyperbolic trigonometric functions $\cosh : \mathbb{C} \longrightarrow \mathbb{C}$ and $\sinh : \mathbb{C} \longrightarrow \mathbb{C}$ given by,

$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$\sinh z = \frac{e^z - e^{-z}}{2}$$

Now, observe that;

$$\cos(iz) = \frac{e^z + e^{-z}}{2} = \cosh(z)$$
$$\sin(iz) = \frac{e^{-z} - e^z}{2i} = i\sinh(z) \implies \sinh(z) = -i\sin(iz)$$

EXERCISE 7. Prove that $\cosh^2 z - \sinh^2 z = 1$