Complex Analysis

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Lecture No – 2.2

Isometries on the Complex Plane

In the last week we defined the field of complex numbers and proved that any complex field will be isomorphic to it. We also defined a metric on the complex numbers using the absolute value, which happened to be the square root of the norm function. And we remarked that the explicit construction of the complex plane should not affect the study of analysis on the field of complex numbers. Thereafter we studied some important topological notions on the complex plane.

In this week we will study some functions on the complex plane and some of its geometric properties. Let us begin this lecture by discussing isometries on the field of the complex numbers.

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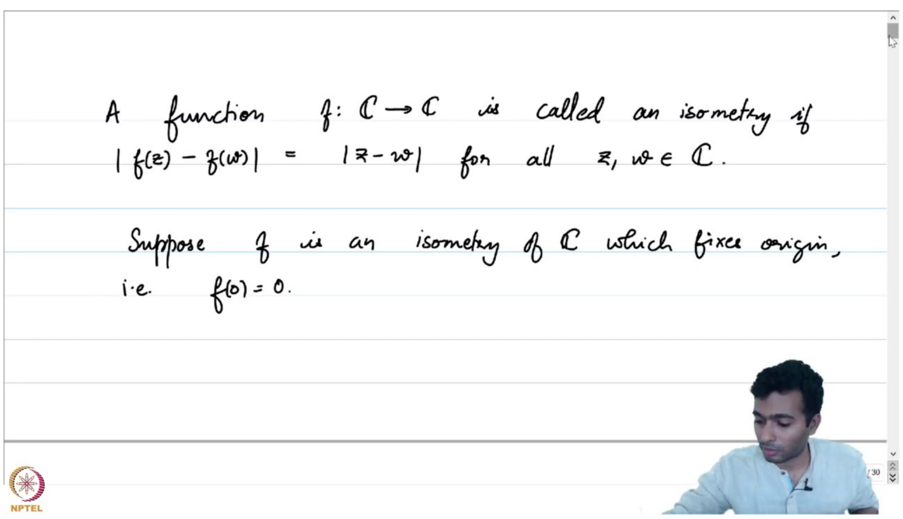

<u>Definition</u>

A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is called an isometry if $|f(z) - f(w)| = |z - w| \; \forall z, w \in \mathbb{C}$.

That is, isometry is a function which preserves the distance between two points.

Claim: Let f be an isometry on $\mathbb{C}$ such that $f(0) = 0$. Then $\langle f(z), f(w)\rangle = \langle z, w\rangle \forall z, w \in \mathbb{C}$.

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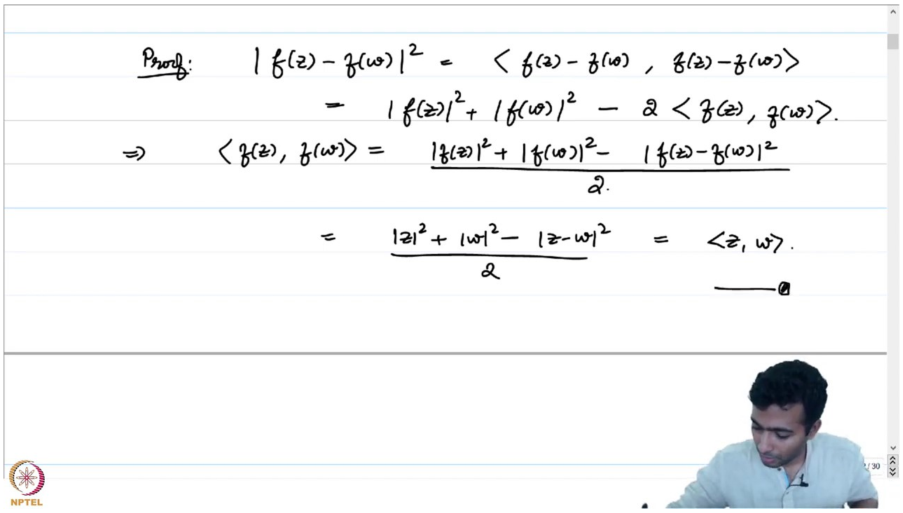
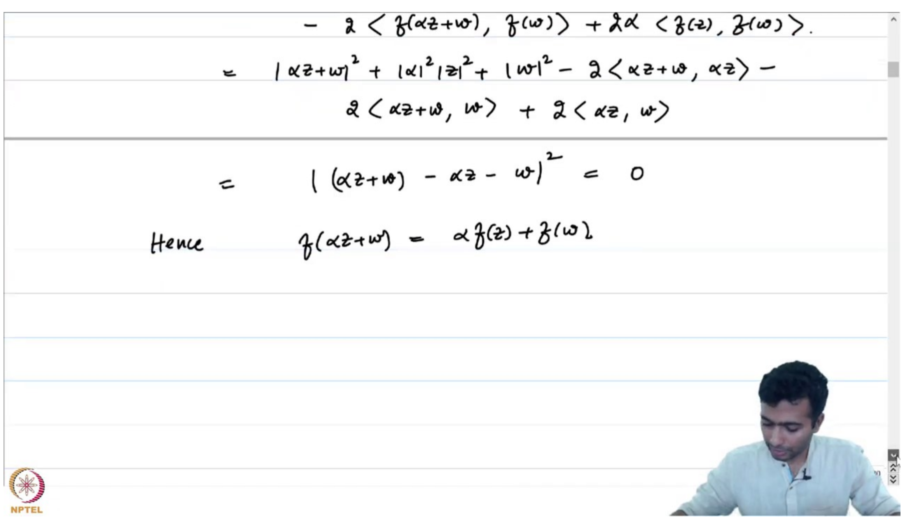


PROOF. $|f(z)-f(w)|^2 = \langle f(z)-f(w), f(z)-f(w)\rangle = |f(z)|^2+|f(w)|^2-2\langle f(z), f(w)\rangle \implies \langle f(z), f(w)\rangle = \frac{|f(z)|^2+|f(w)|^2-|f(z)-f(w)|^2}{2} = \frac{|z|^2+|w|^2-|z-w|^2}{2} = \langle z, w\rangle.$ □

PROPOSITION 1. *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an isometry such that $f(0) = 0$. Then f is a linear map.*

PROOF. **(Refer Slide Time: 09:26)**



For every $\alpha \in \mathbb{R}$,

$|f(\alpha z+w)-\alpha f(z)-f(w)|^2 = \langle f(\alpha z+w)-\alpha f(z)-f(w), f(\alpha z+w)-\alpha f(z)-f(w)\rangle =$
$|f(\alpha z+w)|^2+|\alpha|^2|f(z)|^2+|f(w)|^2-2\alpha\langle f(\alpha z+w), f(z)\rangle-2\langle f(\alpha z+w), f(w)\rangle+2\alpha\langle f(z), f(w)\rangle =$
$|\alpha z+w|^2+|\alpha|^2|z|^2+|w|^2-2\langle \alpha z+w, \alpha z\rangle-2\langle \alpha z+w, w\rangle+2\langle \alpha z, w\rangle = |(\alpha z+w)-\alpha z-w|^2 = 0.$
Hence $f(\alpha z+w) = \alpha f(z)+f(w)$. $\square$

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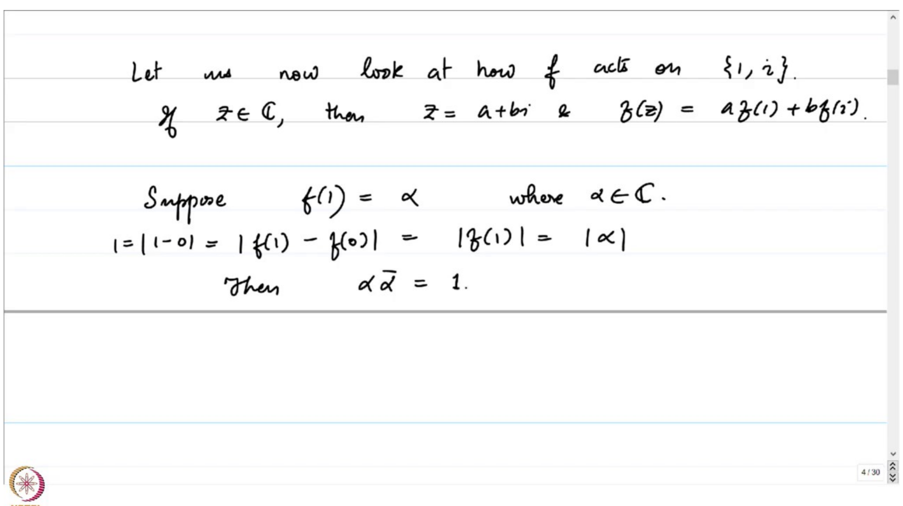

Let us now look at how f acts on $\{1, i\}$. If $z \in \mathbb{C}$, then $z = a+ib$ and $f(z) = af(1)+bf(i)$. Suppose $f(1) = \alpha$, where $\alpha \in \mathbb{C}$. Then, $1 = |1-0| = |f(1)-f(0)| = |f(1)| = |\alpha| \implies \alpha\bar{\alpha} = 1$.

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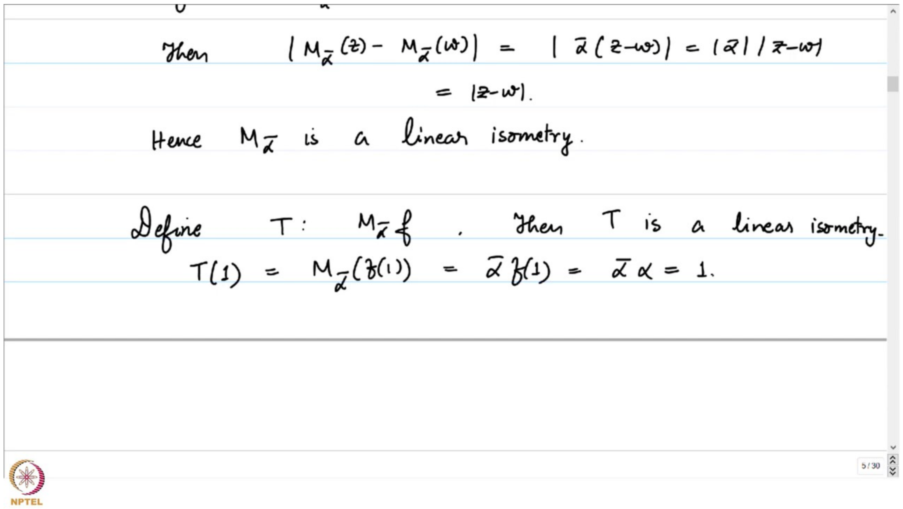

Define $M_{\bar{\alpha}}(w) := \bar{\alpha} w$. Then $|M_{\bar{\alpha}}(z) - M_{\bar{\alpha}}(w)| = |\bar{\alpha}(z-w)| = |\bar{\alpha}||z-w| = |z-w|$. Notice that $M_{\bar{\alpha}}(0) = 0$. Hence $M_{\bar{\alpha}}$ is a linear isometry.

Now reader should verify that the composition of two linear isometry is again a linear isometry.

Define $T := M_{\bar{\alpha}} f$. Then T is a linear isometry. $T(1) = M_{\bar{\alpha}}(f(1)) = \bar{\alpha} f(1) = \bar{\alpha}\alpha = 1$.

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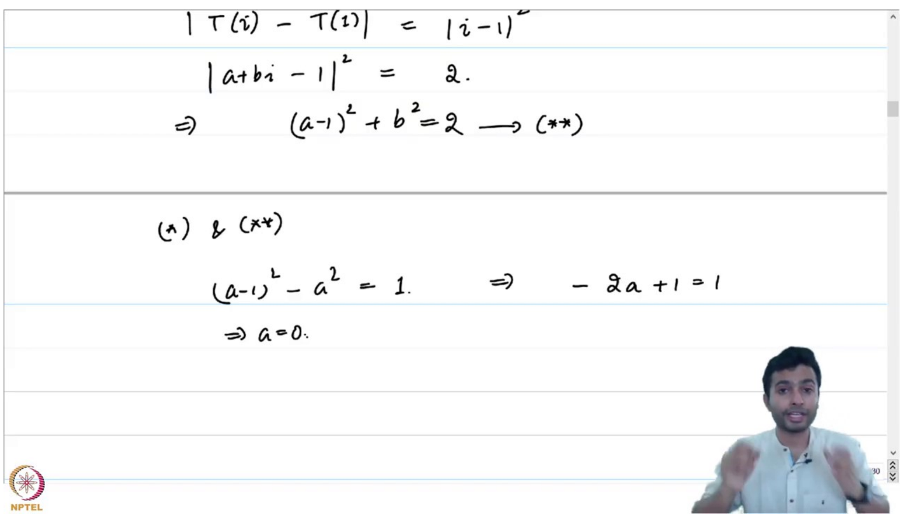

Let $T(i) = a + ib, |T(i) - T(0)| = |i| = 1 \implies |a + ib| = 1 \implies a^2 + b^2 = 1 \longrightarrow (*)$
$|T(i) - T(1)|^2 = |i-1|^2 \implies |a + ib - 1|^2 = 2 \implies (a-1)^2 + b^2 = 2 \longrightarrow (**)$

Now by $(*)$ and $(**)$, $(a-1)^2 - a^2 = 1 \implies -2a+1 = 1 \implies a = 0 \implies b^2 = 1 \implies b = \pm 1$.
Hence, either $T(i) = i$ or $T(i) = -i$. Therefore either $T(z) = z$ or $T(z) = \bar{z}\, \forall z \in \mathbb{C}$.
Now, we have defined $T = M_{\bar{\alpha}} f$, notice that $M_{\bar{\alpha}} M_{\alpha}(w) = \bar{\alpha}(\alpha w) = |\alpha|^2 w = w = M_{\alpha} M_{\bar{\alpha}}(w)$.
Thus either $f(z) = \alpha z\, \forall z \in \mathbb{C}$ or $f(z) = \alpha \bar{z}\, \forall z \in \mathbb{C}$.
Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Then $\alpha \in S^1$.

We have already noticed that, for each element in S^1 we can find an isometry corresponding to that element. That is, if $z_0 \in S^1$,define $f(w) = z_0 w$, then f is an isometry. Observe that the set S^1 can be interpreted geometrically as a unit circle, $S^1 = \{z \in \mathbb{C} : z = a + ib, a^2 + b^2 = 1\}$.

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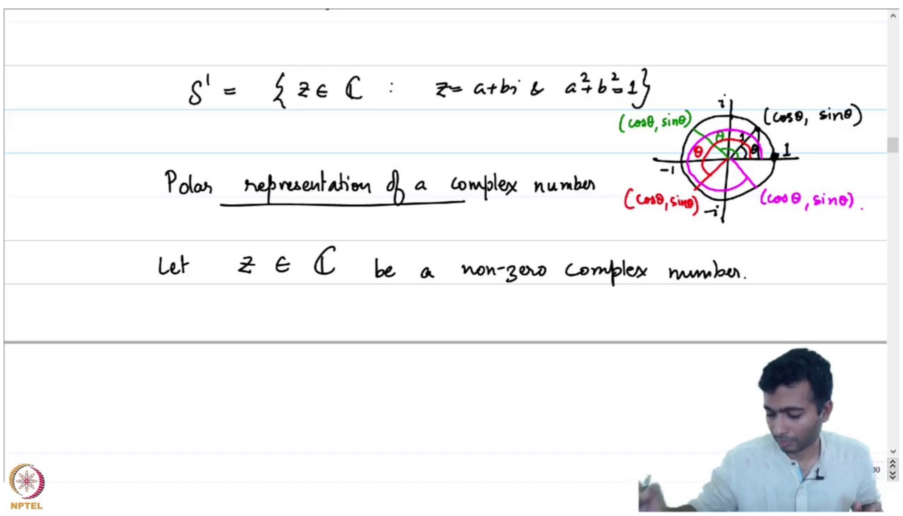

Polar representation of a complex number

Let $z \in \mathbb{C}$ be a non-zero complex number, then $z = |z|\left(\frac{z}{|z|}\right)$. Let $r = |z|, w = \frac{z}{|z|}$, then $r > 0$ and $w \in S^1$. Such a representation of z is unique and is called polar decomposition of z. Since $w \in S^1$, $w = \cos\theta + i\sin\theta$.

Let f be an isometry which 0. Then, let $\alpha \in S^1$ be such that $f(z) = \alpha z$ and $\alpha = \cos\theta + i\sin\theta$.

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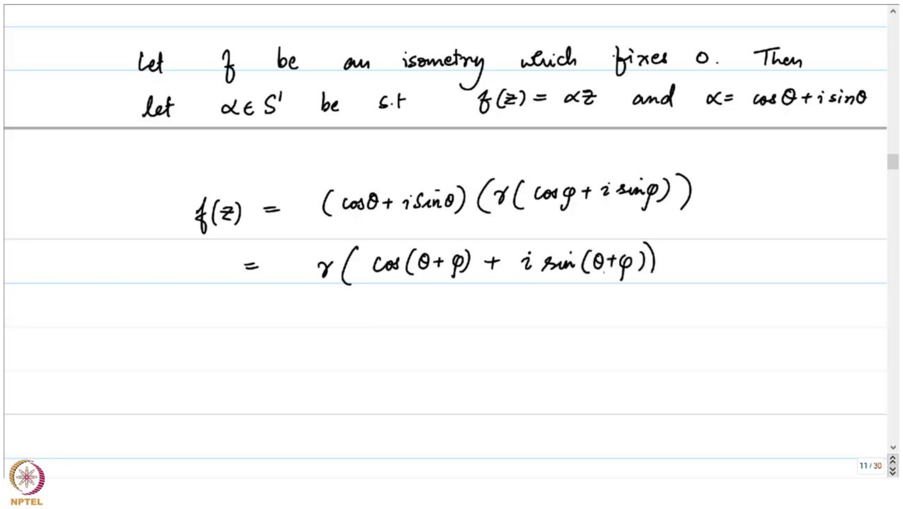

$$f(z) = (\cos\theta + i\sin\theta)(r(\cos\phi + i\sin\phi))$$

$$= r(\cos(\theta+\phi) + i\sin(\theta+\phi)).$$

Then f turns out to be a rotation by θ.

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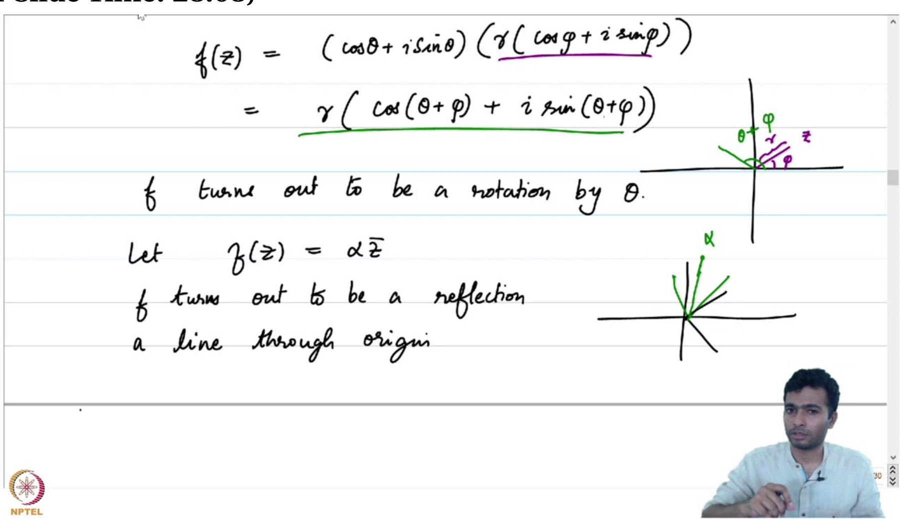

Similarly we can see the case of $f(z) = \alpha\bar{z}$, then f turns out to be a reflection along a line through origin, since on taking the conjugate we are reflecting z along the real line and then we are rotating it.

Polar decomposition is easy when multiplying complex numbers. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then $z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$. That is, geometrically, multiplication of complex numbers is a kind of dilation and rotation.

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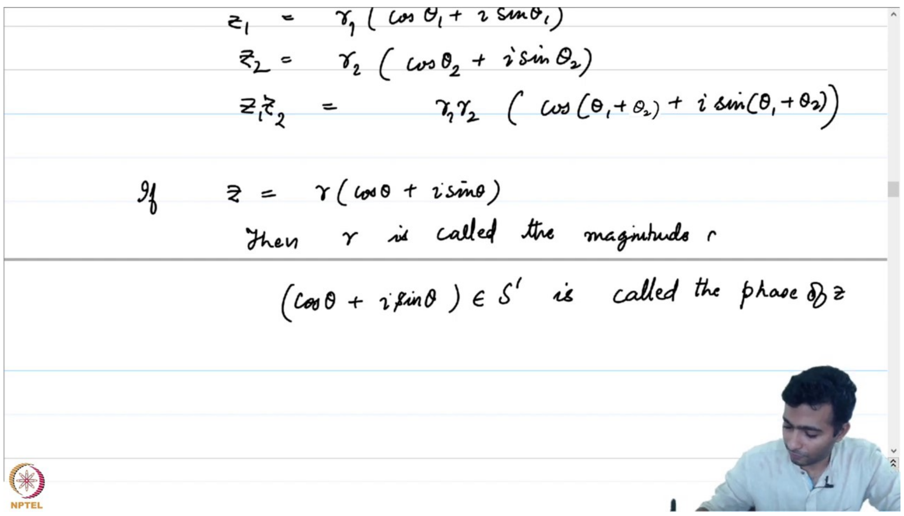

If $z = r(\cos\theta + i\sin\theta)$, then r is called magnitude of z, $(\cos\theta + i\sin\theta) \in S^1$ is called phase of z.

If $\theta \in \mathbb{R}$ such that for $z \in \mathbb{C}$, $z = r(\cos\theta + i\sin\theta)$, then θ is called an argument of z. Notice that, if θ is an argument of z, the basic trigonometric rule tells that, $\theta + 2k\pi$ is also an argument of z, where $k \in \mathbb{Z}$.

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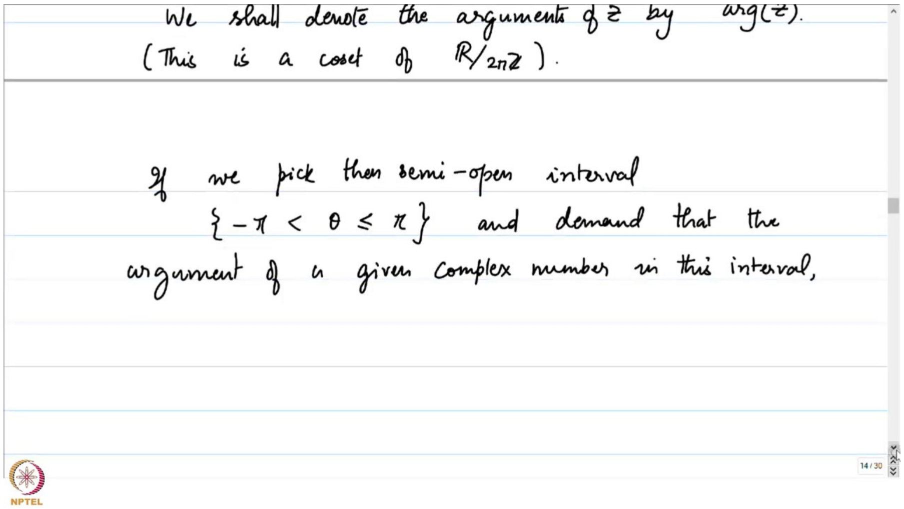

We shall denote argument of z by $\arg(z)$. From the observation we made on argument, in the algebraic point of view, argument is a coset of $\mathbb{R}/2\pi\mathbb{Z}$.

If we pick the semi-open interval $(-\pi.\pi]$ and demand that the argument of given complex number z lies in this interval, then the argument is called the standard argument of z and is denoted by $\mathrm{Arg}(z)$.

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