Complex Analysis

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Lecture – 2.1

Problem Session

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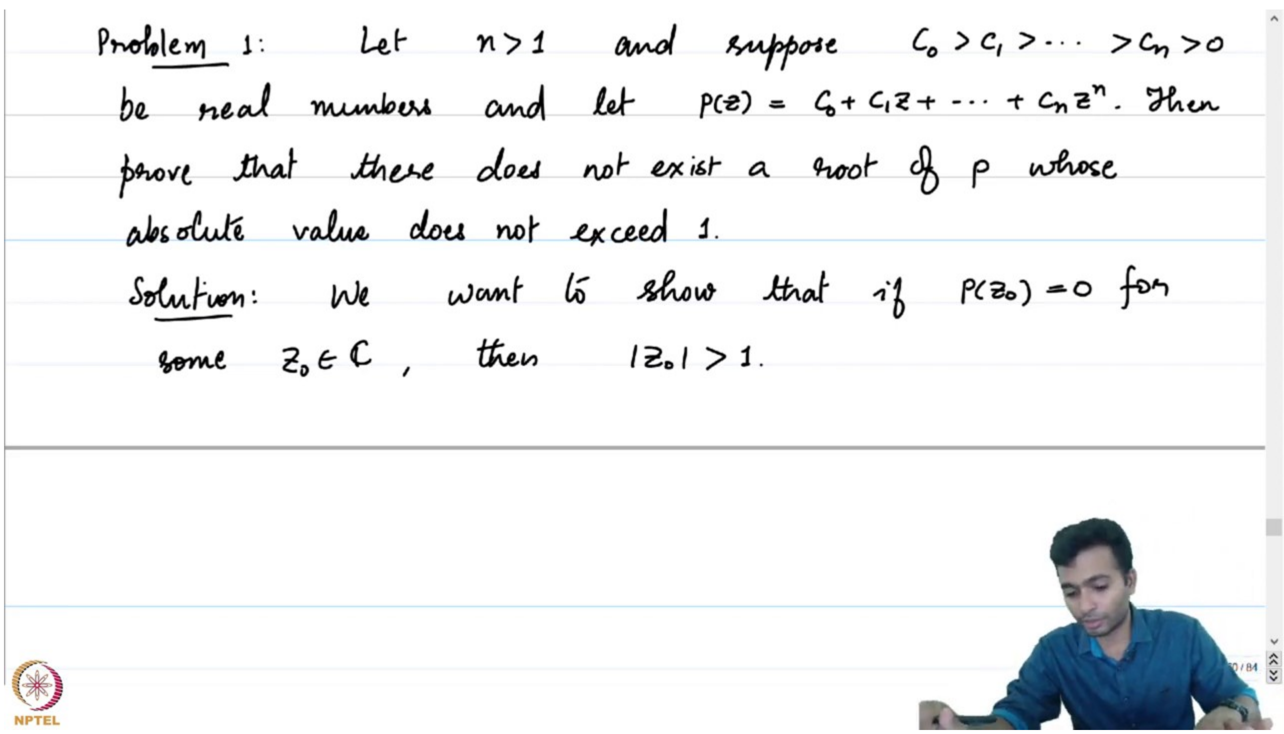

PROBLEM 1. Let $n > 1$ and suppose $c_0 > c_1 > \cdots > c_n > 0$ be real numbers and let $p(z) = c_0 + c_1 z + \cdots + c_n z^n$. Then prove that there does not exist a root of p whose absolute value does not exceed 1.

SOLUTION 1. We want to show that if $p(z_0) = 0$ for some $z_0 \in \mathbb{C}$, then $|z_0| > 1$.

Consider the polynomial $(1-z)p(z)$,

$$
\begin{aligned}
(1-z)p(z) &= (c_0 + c_1 z + \cdots + c_n z^n) - (c_0 z + \cdots + c_n z^{n+1}) \\
&= c_0 + (c_1 - c_0)z + \cdots + (c_n - c_{n-1})z^n - c_n z^{n+1} \\
&= c_0 - \big((c_0 - c_1)z + \cdots + (c_{n-1} - c_n)z^n + c_n z^{n+1}\big).
\end{aligned}
$$

Then note that all the coefficients of the terms inside the parentheses are positive numbers.

Let $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$. Then we want to prove that $|z_0| > 1$. We may consider the case when z_0 is in the unit disc, $|z_0| \leq 1$

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Let $z_0 \in \mathbb{C}$ s.t $p(z_0) = 0$

$$|c_0| = |(c_0 - c_1)z_0 + \cdots + (c_{n-1} - c_n)z_0^n + c_n z_0^{n+1}|$$

$$= |z_0| \left(| (c_0 - c_1) + \cdots + (c_{n-1} - c_n) z^{n-1} + c_n z^n | \right)$$

$$\leq |z_0| \left(|c_0 - c_1| + |(c_2 - c_1) z_0| + \cdots + |(c_{n-1} - c_n) z_0^{n-1}| + |c_n z_0^n| \right.$$

$$\leq |z_0| \left(c_0 - c_1 + (c_2 - c_1) \right.$$

$$
\begin{aligned}
c_0 = |c_o| &= |(c_0 - c_1)z_0 + \cdots + (c_{n-1} - c_n)z_0^n + c_n z_0^{n+1}| \\
&= |z_0| \left(|(c_0 - c_1) + \cdots + (c_{n-1} - c_n)z_0^{n-1} + c_n z^n| \right) \\
&\leq |z_0| \left(|(c_0 - c_1)| + \cdots + |(c_{n-1} - c_n)z_0^{n-1}| + |c_n z^n| \right) \\
&= |z_0| \left((c_0 - c_1) + \cdots + (c_{n-1} - c_n)|z_0^{n-1}| + c_n |z^n| \right) \\
&\leq (c_0 - c_1) + (c_2 - c_1) + \cdots + (c_{n-1} - c_n) + c_n \\
&= c_0
\end{aligned}
$$

Since we started with c_0 and ended with c_0, all the inequalities in the above equations will be equalities.

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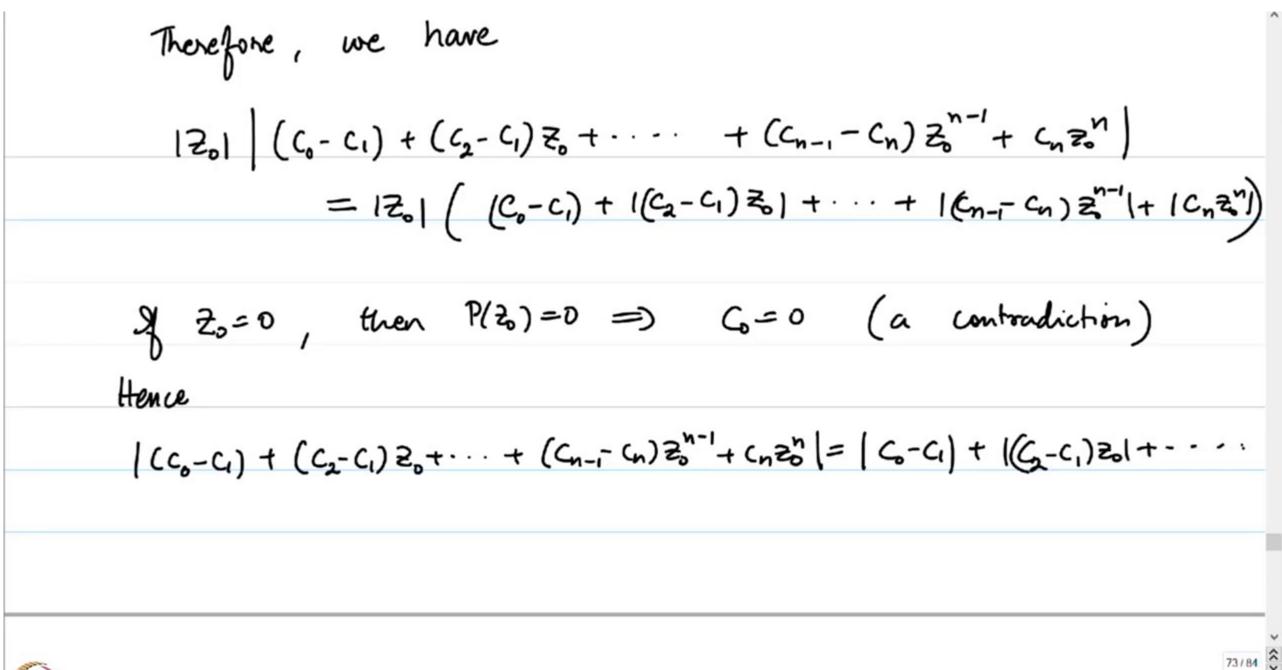

Therefore, we have

$$|z_0|\left(|(c_0-c_1)+\cdots+(c_{n-1}-c_n)z_0^{n-1}+c_nz^n|\right)=|z_0|\left(|(c_0-c_1)|+\cdots+|(c_{n-1}-c_n)z_0^{n-1}|+|c_nz^n|\right)$$

If $z_0=0$, then $p(z_0)=0 \implies c_0=0$, which is a contradiction. Hence $z_0\neq 0 \implies$

$$\left(|(c_0-c_1)+\cdots+(c_{n-1}-c_n)z_0^{n-1}+c_nz^n|\right)=\left(|(c_0-c_1)|+\cdots+|(c_{n-1}-c_n)z_0^{n-1}|+|c_nz^n|\right)$$

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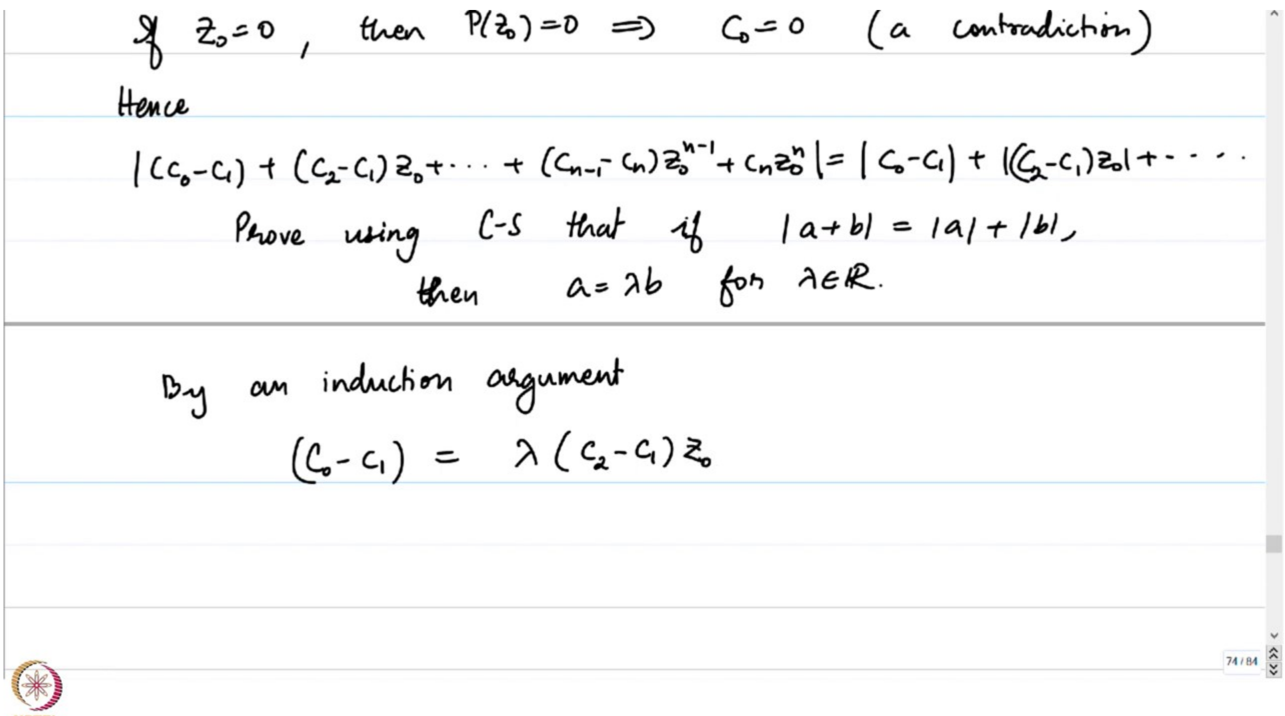

Reader should verify using Cauchy–Schwarz inequality that if $|a+b|=|a|+|b|$, then $a=\lambda b$ for $\lambda\in\mathbb{R}$.

Then, by an induction argument that reader should verify,

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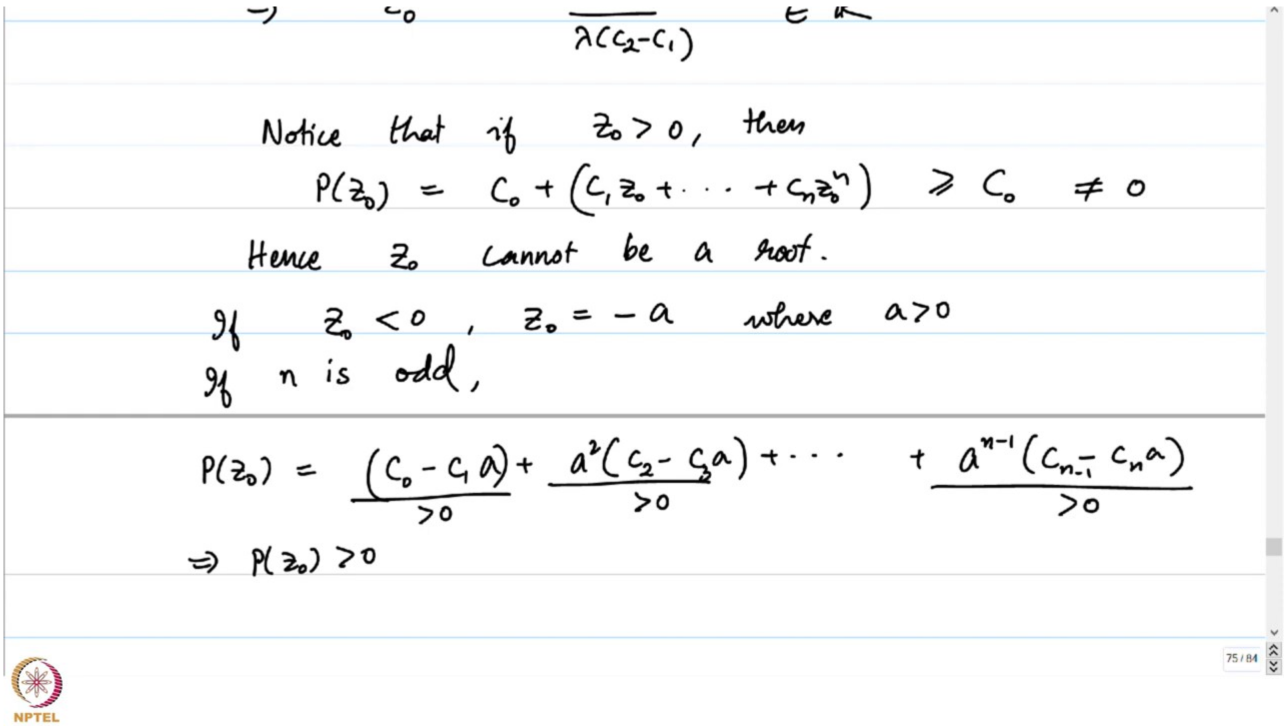

for $\lambda \in \mathbb{R}$, $(c_0 - c_1) = \lambda(c_2 - c_1)z_0 \implies z_0 = \frac{c_0 - c_1}{\lambda(c_2 - c_1)} \in \mathbb{R}$.

Note that if $z_0 > 0$, then $p(z_0) = c_0 + c_1 z_0 + \cdots + c_n z_0^n \geq c_0 \neq 0$. Hence z_0 cannot be root.

If $z_0 < 0, z_0 = -a, a > 0$.

If n is odd, $p(z_0) = (c_0 - c_1 a) + a^2(c_2 - c_3 a) + \cdots + a^{n-1}(c_{n-1} - c_n a)$, then each term inside the parentheses of the sum on RHS is positive. Hence in this case also $p(z_0) > 0$.

If n is even, $p(z_0) = (c_0 - c_1 a) + a^2(c_2 - c_3 a) + \cdots + a^{n-2}(c_{n-2} - c_{n-1} a) + a^n c_n$, then here also each term inside the parentheses of the sum on RHS is positive $\implies p(z_0) \neq 0$.

Hence we have proved that if $|z_0| \leq 1 \implies p(z_0) \neq 0$.

PROBLEM 2. Let $z, w \in \mathbb{C}$ such that $(1 + |z|^2)w = (1 + |w|^2)z$. Then either $z = w$ or $z\overline{w} = 1$.

SOLUTION 2. We know that $|z|^2 = z\overline{z}$. Then,

$$(1+|z|^2)w = (1+|w|^2)z$$

$$w + z\overline{z}w = z + w\overline{w}z$$

$$w(1-\overline{w}z) = z(1-\overline{z}w).$$

If $z = 0 \implies w = o$, hence $z = w$.

Suppose $z \neq 0$ and $z\overline{w} \neq 1$, then
$\frac{w}{z} = \frac{(1-\overline{z}w)}{(1-\overline{w}z)} \implies |\frac{w}{z}| = |\frac{1-\overline{z}w}{1-\overline{w}z}| = \frac{|1-\overline{z}w|}{|1-\overline{w}z|} = 1 \implies |w| = |z| \implies (1+|w|^2) = (1+|z|^2)$.
Hence $(1+|w|^2)z = (1+|z|^2)w \implies z = w$.

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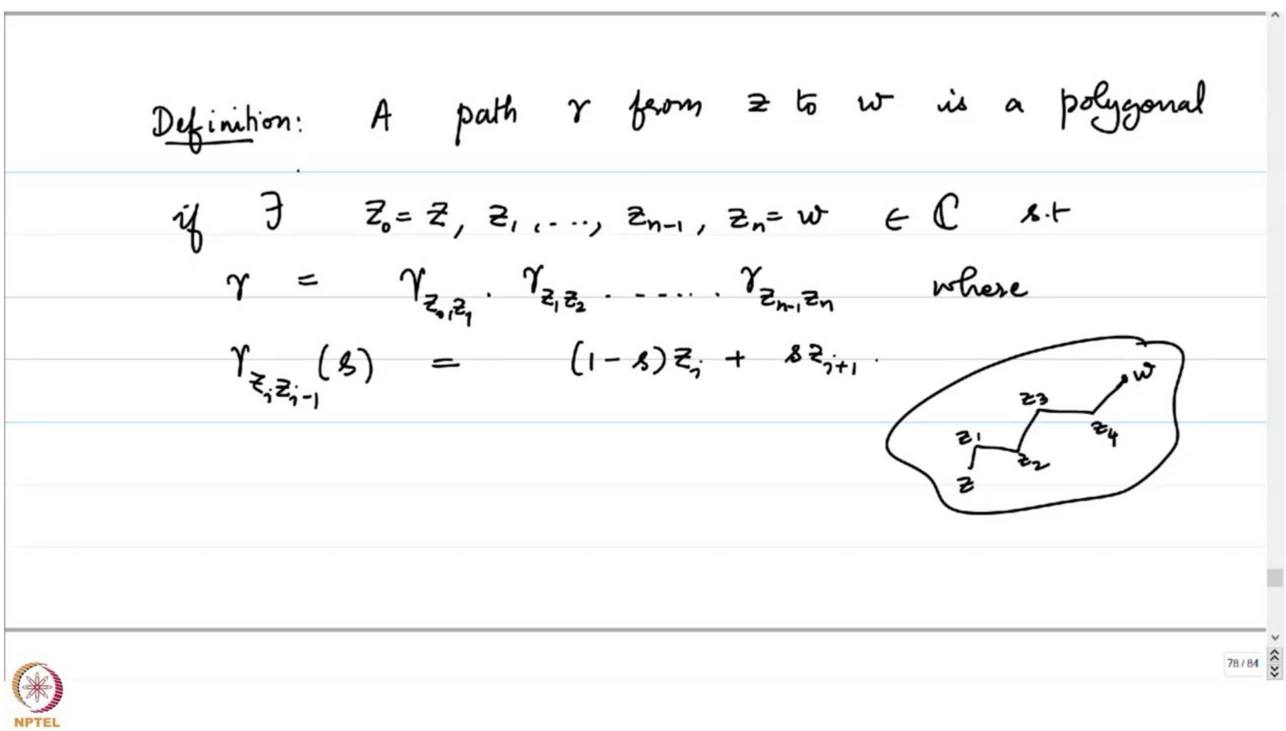

DEFINITION 3. A path γ from z to w is a polygonal if $\exists\ z = z_0, z_1, \ldots z_{n-1}, z_n = w \in \mathbb{C}$ such that $\gamma = \gamma_{z_0,z_1} \cdot \gamma_{z_1,z_2} \cdots \gamma_{z_{n-1},z_n}$ where $\gamma_{z_i,z_{i+1}}(s) = (1-s)z_i + sz_{i+1}$.

That is polygonal path between two points in the complex plane is obtained by concatenating straight line paths between finite number of points.

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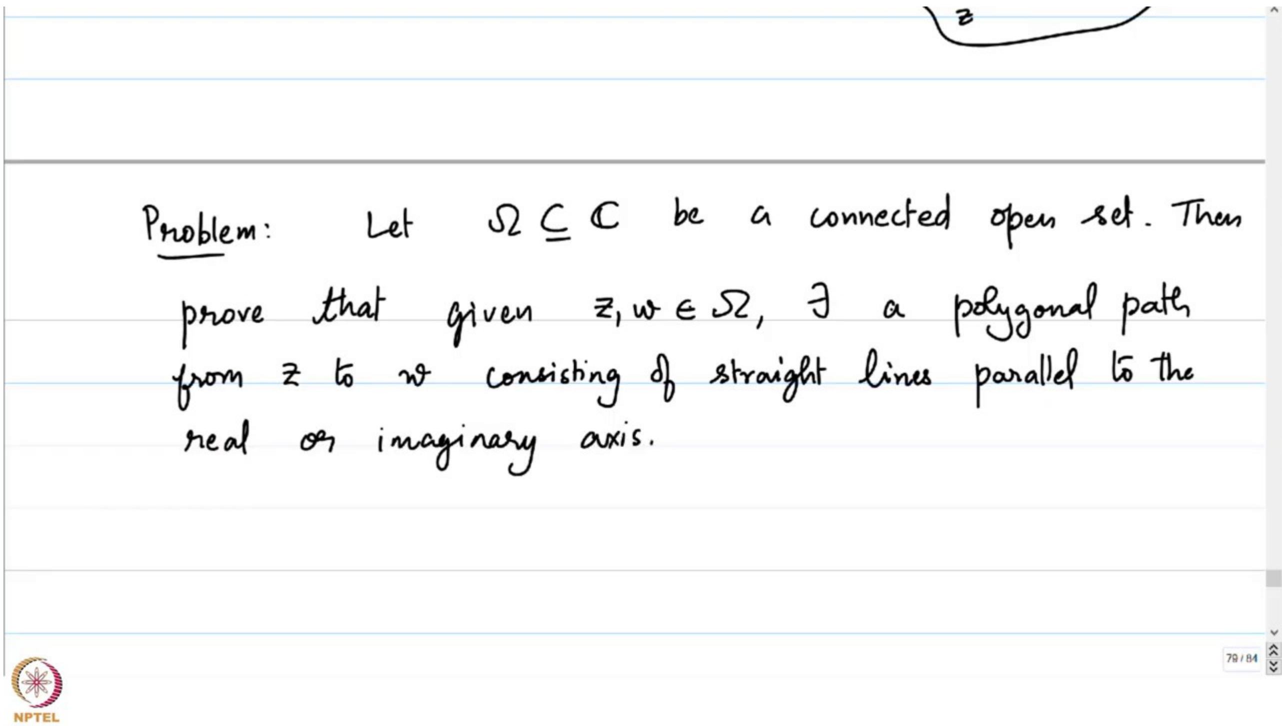

PROBLEM 3. Let $\Omega \subseteq \mathbb{C}$ be a connected open set. Then prove that given $z, w \in \Omega, \exists$ a polygonal path from z to w consisting of straight lines parallel to the real or imaginary axis.

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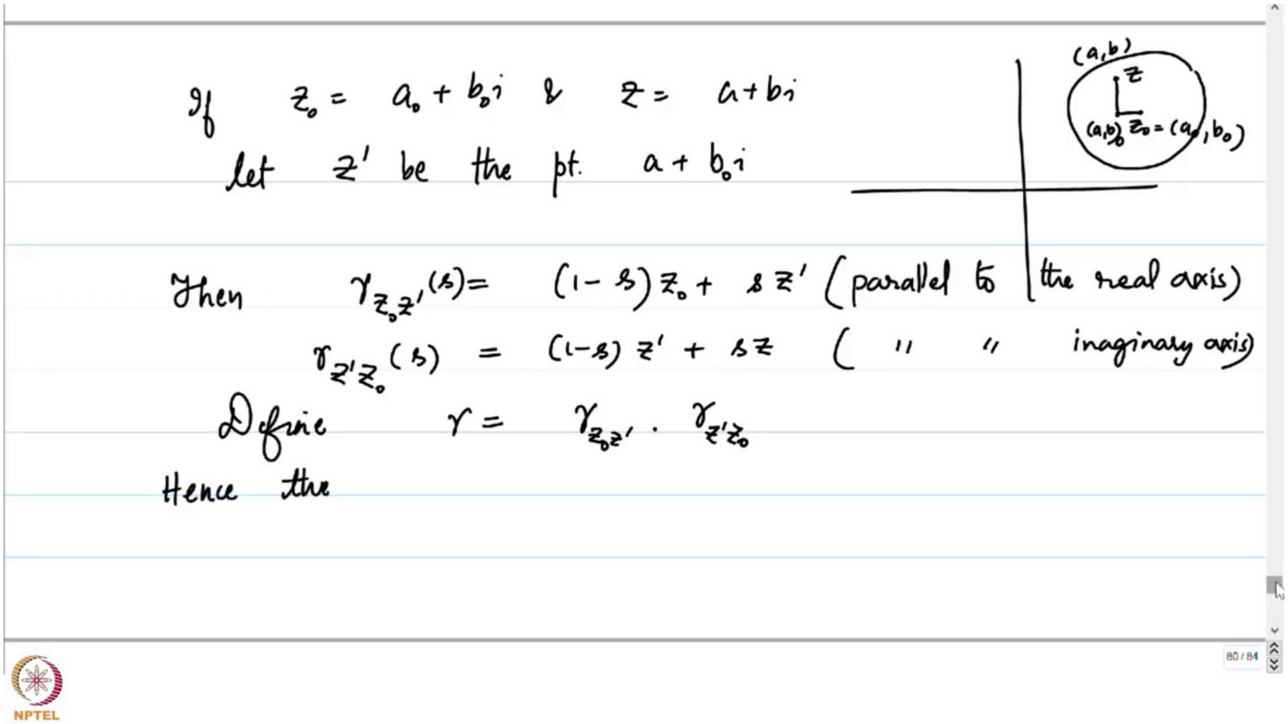

SOLUTION 4. Suppose $z_0 \in \mathbb{C}$ and $r > 0$. Then we shall first prove that $D(z_0, r)$ satisfies the given condition.

If $z_0' = a_0 + ib_0$ and $z = a + ib$ be two points inside $D(z_0, r)$. Let $z'' = a + ib_0 \in D(z_0, r)$. (Reader is strongly suggested to draw a picture with given details to get a good idea.)

$\gamma_{z_0', z''}(s) = (1-s)z_0' + sz''$(paralell to real axis)

$\gamma_{z'', z}(s) = (1-s)z'' + sz$(paralell to imaginary axis)

Define $\gamma = \gamma_{z_0', z''} \cdot \gamma_{z'', z}$, then γ will be a polygonal path from z_0' to z with desired condition.

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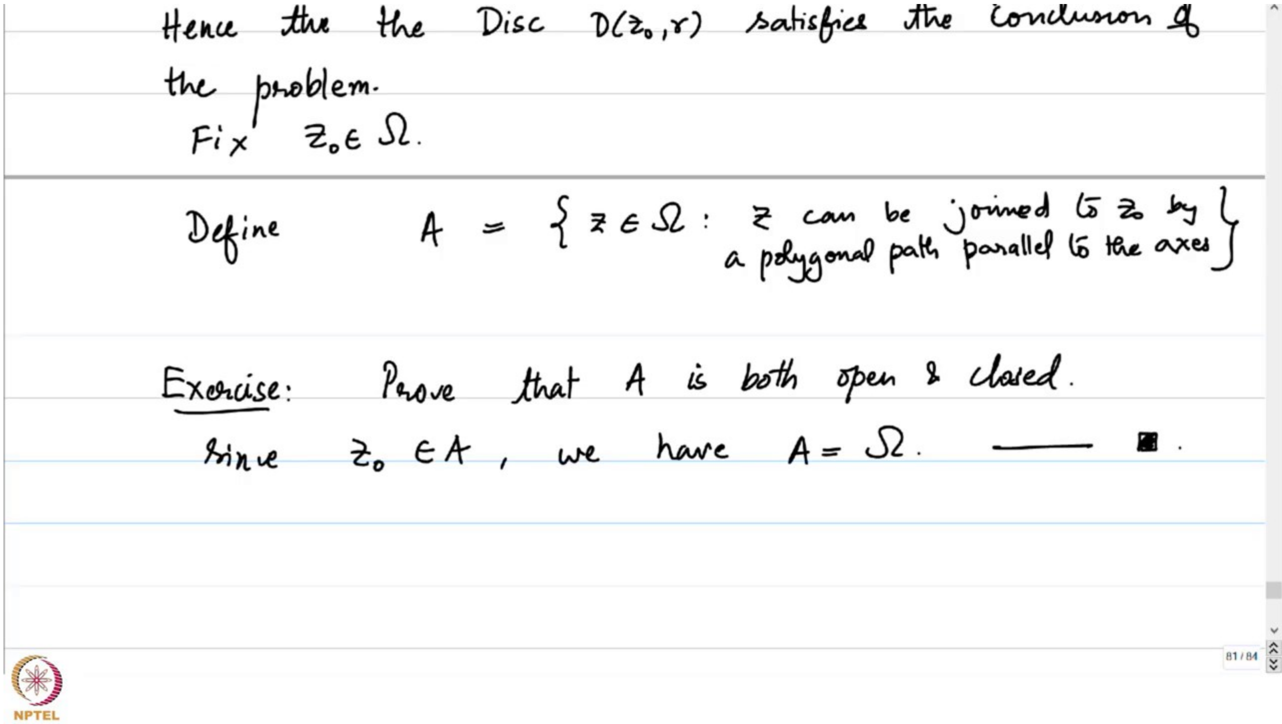

Hence the disc $D(z_0, r)$ satisfies the condition of the problem.

Now, fix $z_0 \in \Omega$, define $A = \{z \in \Omega : z$ can be joined to z_0 by a polygonal path parallel to axes$\}$. Now it is left to reader to verify that A is both open and closed. Since $z_0 \in A \implies A \neq \varnothing \implies A = \Omega$, as Ω is connected.

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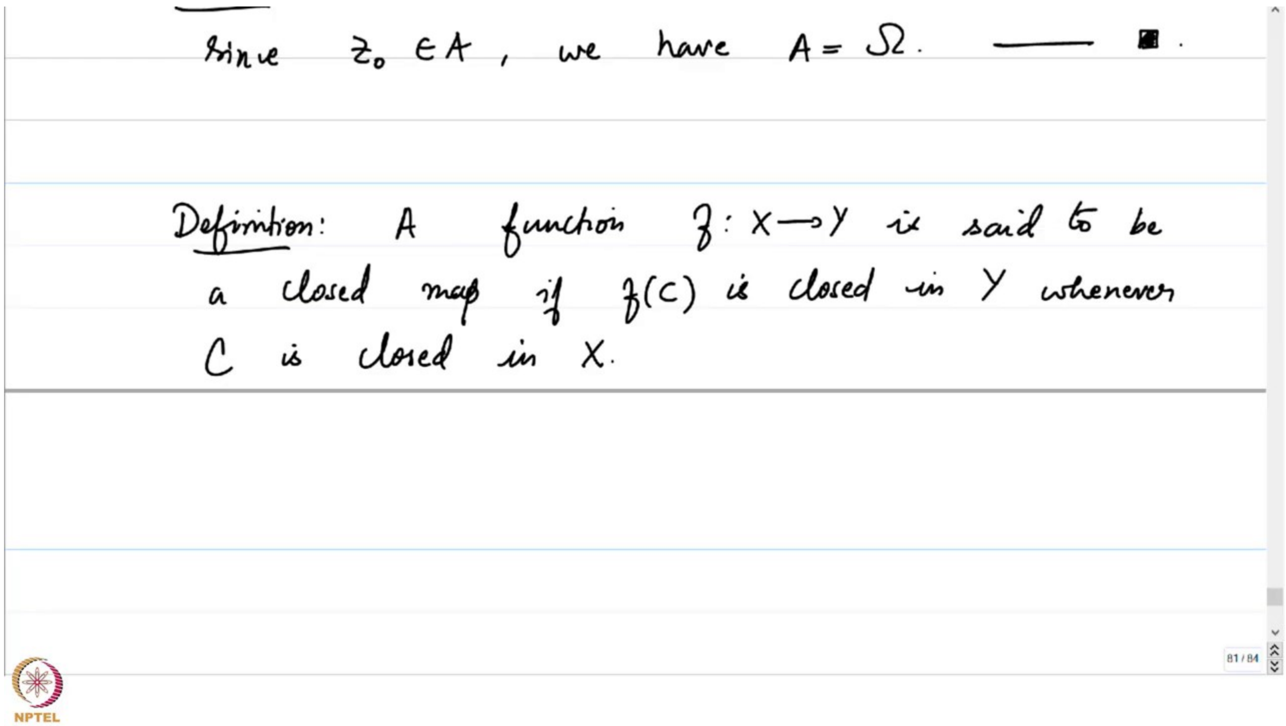

DEFINITION 5. Let X, Y be metric spaces. A function $f : X \longrightarrow Y$ is said to be a closed map if $f(C)$ is closed in Y whenever C is closed in X.

PROBLEM 4. Let $f : X \longrightarrow Y$ be a continuous mapping from a compact metric space to another metric space. Then f is a closed map.

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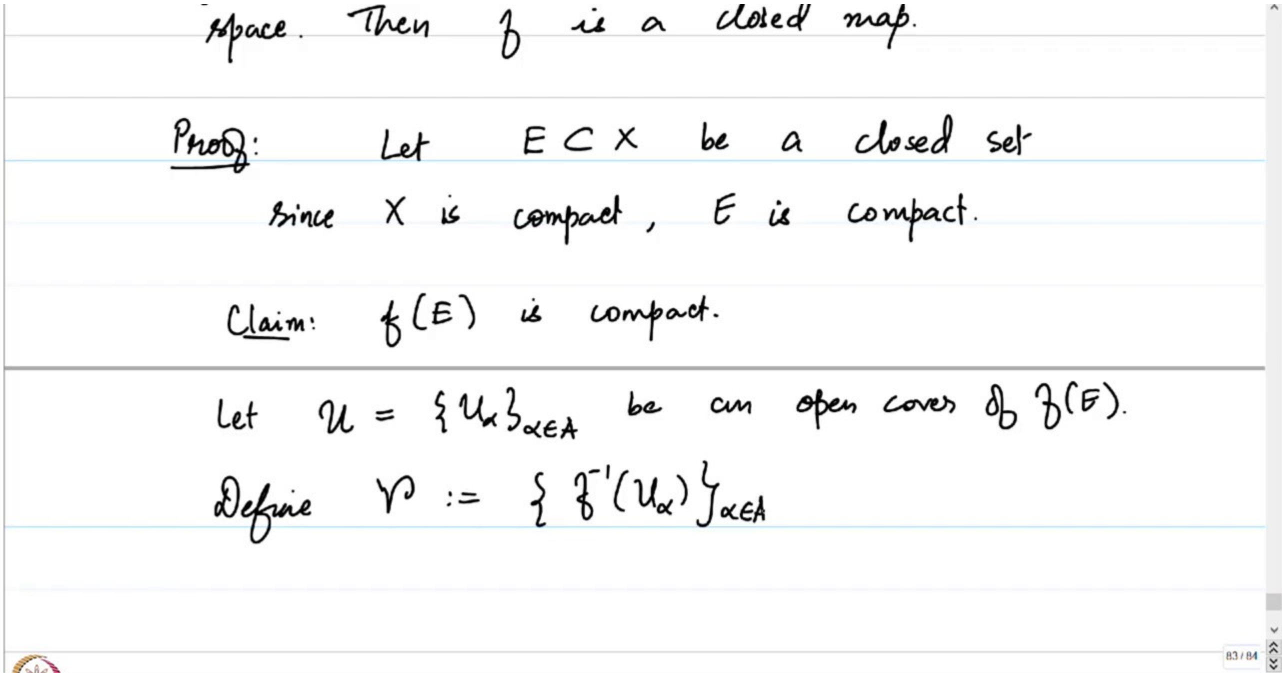

SOLUTION 6. Let $E \subset X$ be a closed subset of X. Since X is compact, E is compact.

Claim: $f(E)$ is compact.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of $f(E)$. Define $\mathcal{V} := \{f^{-1}(U_\alpha)\}_{\alpha\in A}$, then $\mathcal{V}$ is an open cover of E.

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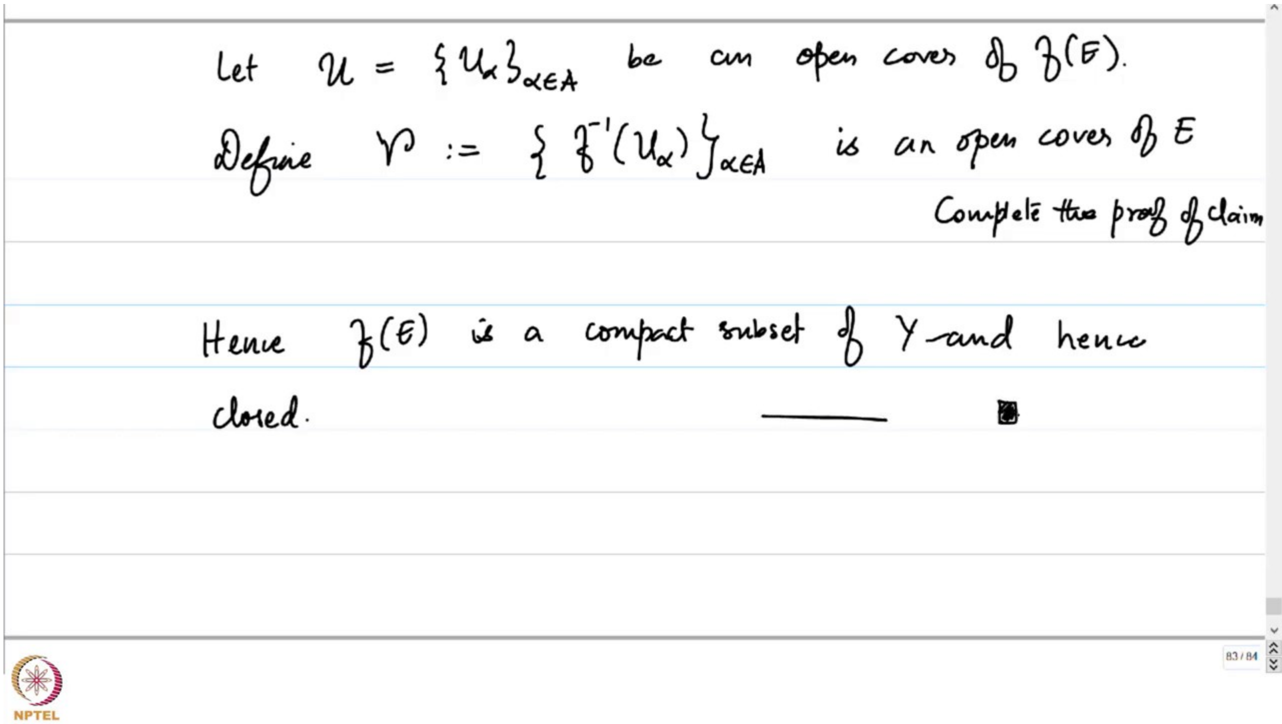

Now it is left as an exercise to reader to conclude that $f(E)$ is also compact. Then $f(E)$ is compact in $Y \implies f(E)$ is closed.

THEOREM 5 (**Heine-Borel**). *A subset $K \subseteq \mathbb{R}^n$ is compact if and only if K is closed and bounded in $\mathbb{R}^n$.*

PROOF. ($\Rightarrow$) Let K be compact, then K is closed. Consider $\mathscr{U} = (B(0,n))_{n\in\mathbb{N}}$. Then $\mathscr{U}$ is an open cover of K. Since K is compact, $\exists\, n \in \mathbb{N}$ such that

$$K \subset B(0,1) \cup \cdots \cup B(0,n)$$

. Hence K is bounded.

($\Leftarrow$) Assume that K is closed and bounded.

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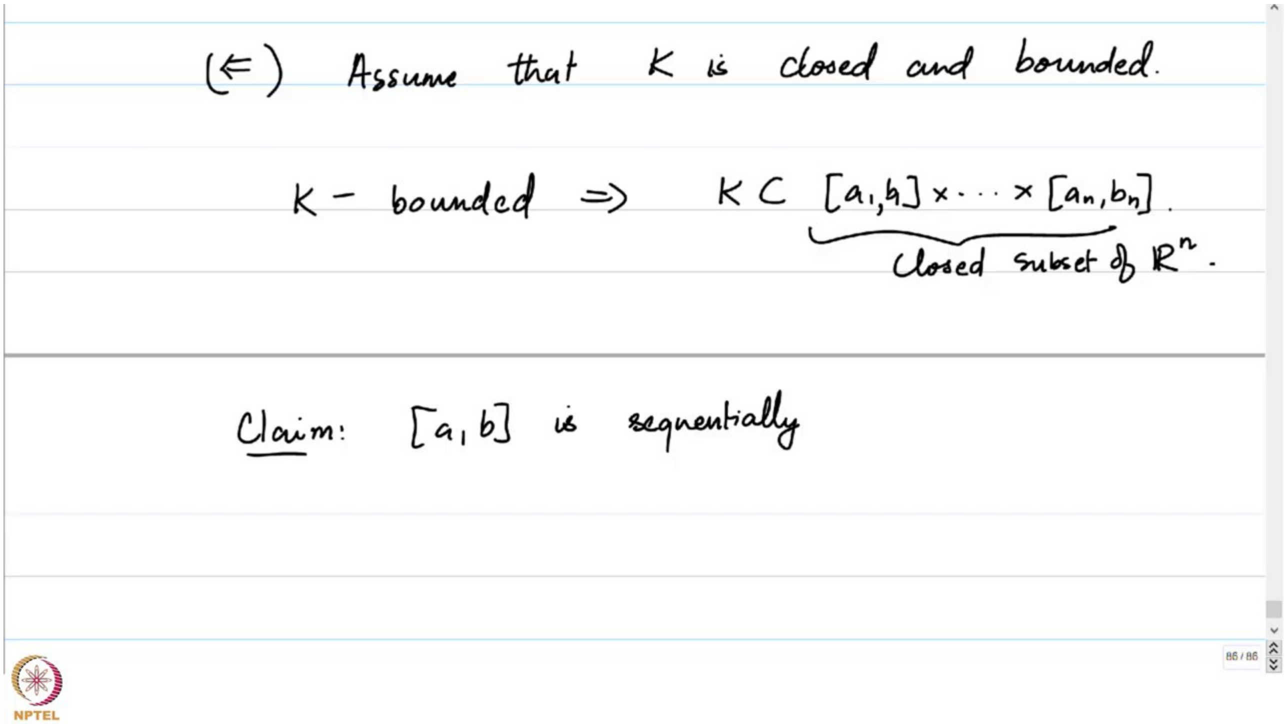

K is bounded $\implies K \subset [a_1, b_1] \times \cdots \times [a_n, b_n]$.

Claim: $[a, b]$ is sequentially compact.

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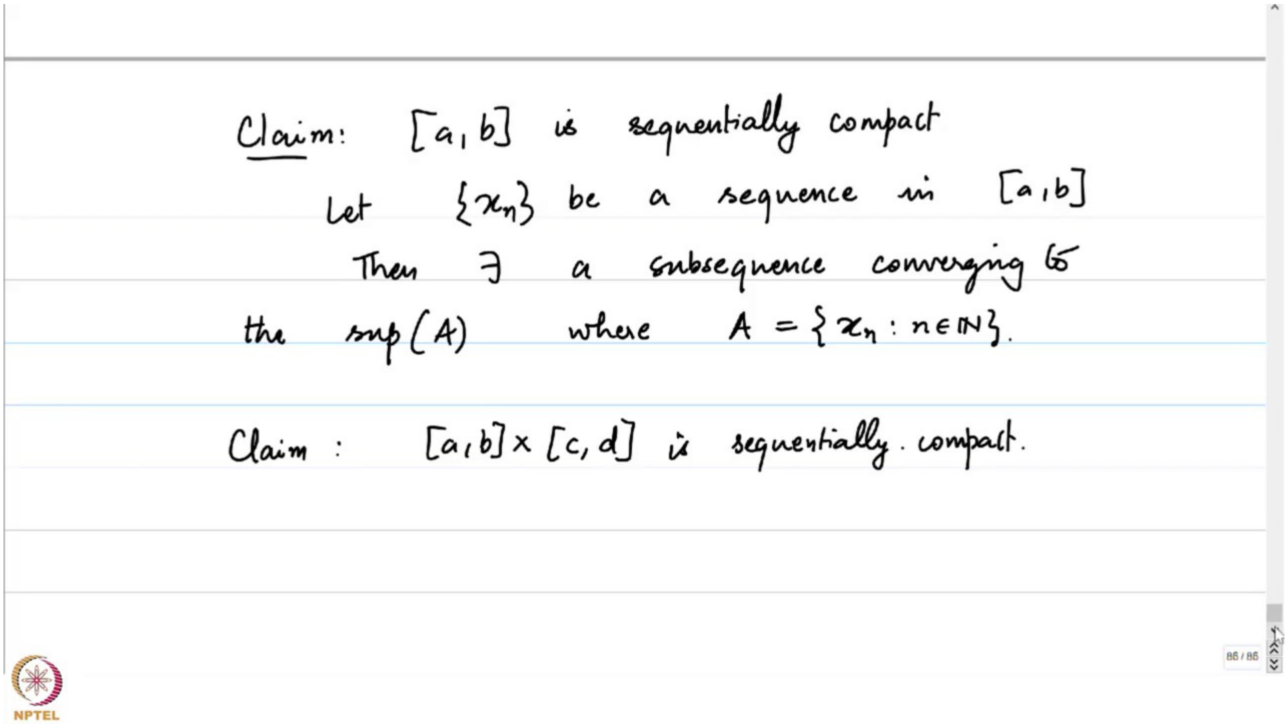

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $[a,b]$. Let $A = \{x_n : n \in \mathbb{N}\}$, then $\exists$ a subsequence converging to $sup(A)$ by using Bolzano- Weierstrass theorem. Since $[a,b]$ is a closed set, supremum will belong $[a,b]$. Hence $[a,b]$ is sequentially compact.

Claim: $[a,b] \times [c,d]$ is sequentially compact.

Let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a sequence in $[a,b] \times [c,d]$. Let $\{x_{n_k}\}_{k\in\mathbb{N}}$ be a convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$ in $[a,b]$, whose existence is obtained by above claim. Similarly, let $\{y_{n_k}\}_{k\in\mathbb{N}}$ be a convergent subsequence of $\{y_n\}_{n\in\mathbb{N}}$ in $[c,d]$.

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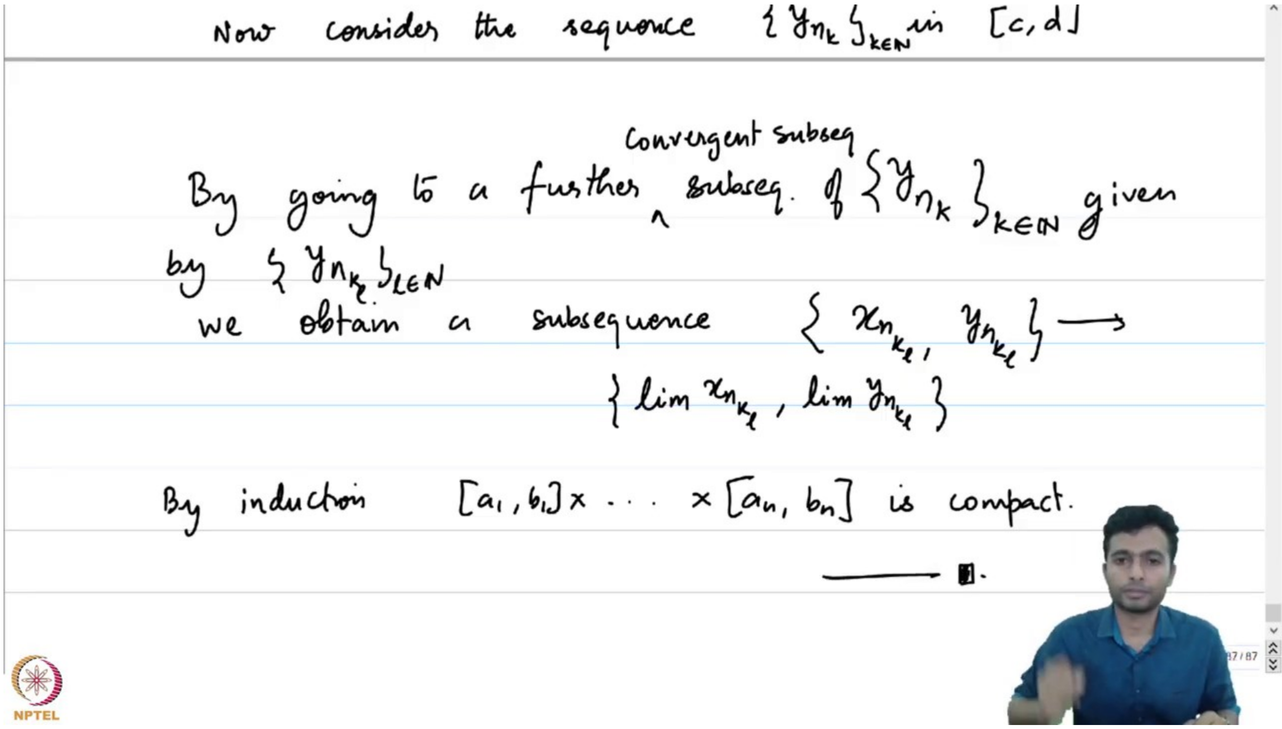

Now we obtain a subsequence $\{(x_{n_k}, y_{n_k})\}_{k\in\mathbb{N}}$ of $\{(x_n, y_n)\}_{n\in\mathbb{N}}$. Hence $[a,b]\times[c,d]$ is also sequentially compact.

By induction $[a_1,b_1]\times\cdots\times[a_n,b_n]$ is also sequentially compact. We know that in a metric space sequentially compactness coincides with compactness, hence $[a_1,b_1]\times\cdots\times[a_n,b_n]$ is compact set. Since K is a closed subset of a compact set, K is also compact.

$\square$