Complex Analysis

Prof. Pranav Haridas

Kerala School of Mathematics

Lecture No – 48

Little Picard's Theorem

The little Picard's theorem tells us that if you have a entire function which is non-constant then it cannot omit 2 points. We have developed all the machinery needed to prove the theorem, but we will first give a characterization of holomorphic functions defined on a simply connected domain which omits 2 points.

PROPOSITION 1. *Let* Ω *be an open connected subset of* $\mathbb{C}$ *which is simply connected. Let* $f:\Omega\longrightarrow\mathbb{C}$ *be a holomorphic function which omits* 0 *and* 1*. Then there exists a holomorphic function* $g:\Omega\longrightarrow\mathbb{C}$ *such that*

$$f(z) = -\exp\left(\pi i \cosh\left(2g(z)\right)\right).$$

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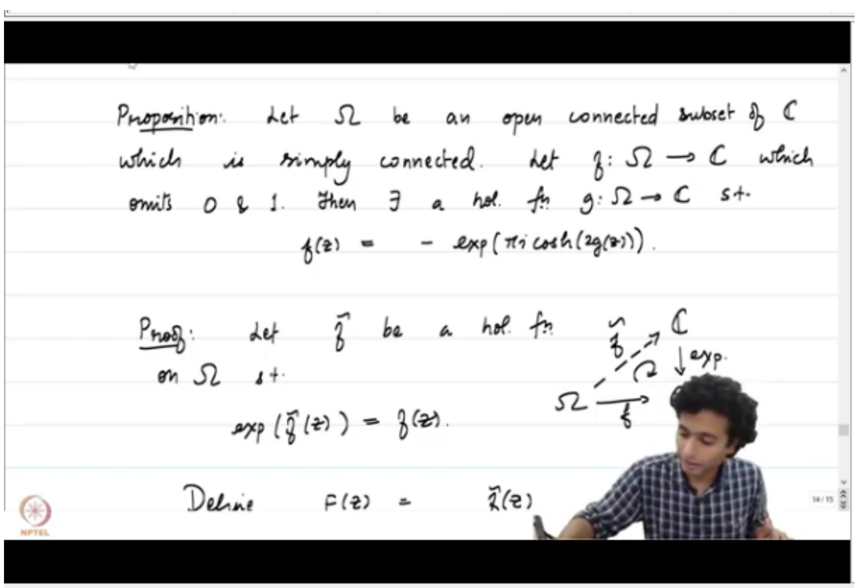

PROOF. Since $f : \Omega \longrightarrow \mathbb{C}^*$, we have a lift of f with respect to exp. Let $\tilde{f}$ be a holomorphic function on Ω such that $\exp\left(\tilde{f}(z)\right) = f(z)$. Define $F(z) = \dfrac{\tilde{f}(z)}{2\pi i}$.

If $F(z_0) = n$ for some $z_0 \in \Omega$, then

$$f(z_0) = \exp\left(\tilde{f}(z_0)\right) = \exp(2\pi i n) = 1$$

which is a contradiction as $f(\Omega) \subseteq \mathbb{C} \setminus \{0,1\}$. Hence F does not take integer values.

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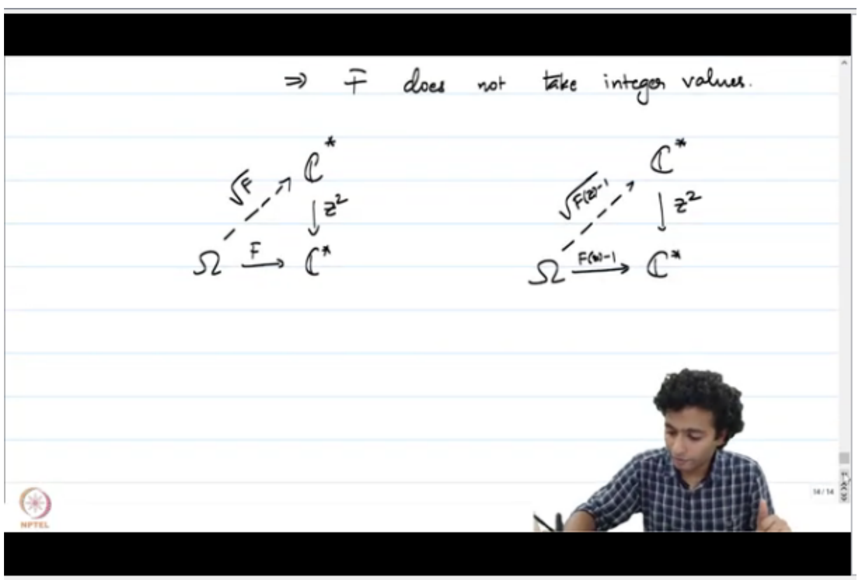


Now, consider the covering map $p : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ given by $p(z) = z^2$. Then we have a lift $\sqrt{F}$ of F with respect to p, $\Omega \xrightarrow{\sqrt{F}} \mathbb{C}^*$. Also since F does not take integer values, $F(z) - 1$ does not vanish and is also holomorphic. Hence we have a lift $\sqrt{F(z)-1}$ of $F(z) - 1$ with respect to p, $\Omega \xrightarrow{\sqrt{F(z)-1}} \mathbb{C}^*$.

Define $H : \Omega \longrightarrow \mathbb{C}$ given by $H(z) = \sqrt{F(z)} - \sqrt{F(z)-1}$. Notice that $H(z) \neq 0$ for every $z \in \Omega$. That is, $H : \Omega \longrightarrow \mathbb{C}^*$.

Then we have a lift g of H with respect to exp. That is $\exp\big(g(z)\big) = H(z)$. Note that

$$\begin{aligned}\cosh\big(2g(z)\big)+1 &= \frac{e^{2g(z)}+e^{-2g(z)}}{2}+1\\ &= \frac{\left(e^{g(z)}+e^{-g(z)}\right)^2}{2}\\ \implies \cosh\big(2g(z)\big)+1 &= \frac{\left(H(z)+\frac{1}{H(z)}\right)^2}{2}\\ &= 2F(z).\end{aligned}$$

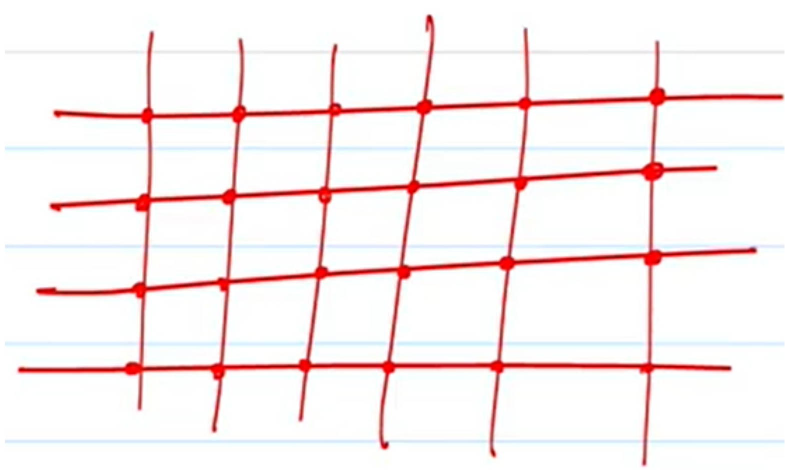
Thus,

$$f(z) = e^{\tilde{f}(z)} = e^{2\pi i F(z)} = e^{\pi i \cosh(2g(z))}e^{\pi i} = -e^{\pi i \cosh(2g(z))}.$$

□

PROPOSITION 2. *The function g in the Proposition 1 does not contain any disk of radius* 1 *in its image.*

PROOF. Let $S = \left\{\pm\ln\left(\sqrt{n}+\sqrt{n-1}\right)+\frac{1}{2}im\pi : n \geq 1, m \in \mathbb{Z}\right\}$. Then notice that the points in the set S can be represented in complex plane as the darkened dots in the following grid.



Claim: $g(\Omega) \cap S = \varnothing$.

Assume that claim is true.

The height of each rectangle can be bounded by

$$\left|\frac{i}{2}(m+1)\pi - \frac{i}{2}m\pi\right| = \frac{\pi}{2} < \sqrt{3}.$$

Also, if we define $\phi(x) = \ln\left(\sqrt{x+1}+\sqrt{x}\right) - \ln\left(\sqrt{x}+\sqrt{x-1}\right)$, then one can verify that ϕ is a decreasing function and $\phi(1) < 1$. Thus, the width of each rectangle can be bounded by

$$\left|\ln\left(\sqrt{n+1}+\sqrt{n}\right) - \ln\left(\sqrt{n}+\sqrt{n-1}\right)\right| < 1.$$

Hence the diagonal of each rectangle is bounded by $\sqrt{\sqrt{3}^2+1} = 2$.

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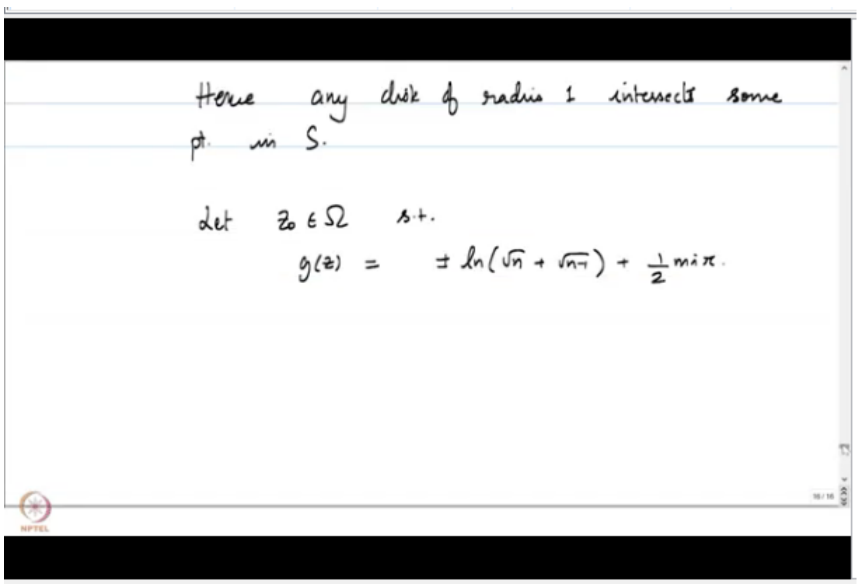

Hence any disk of radius 1 intersects some point in S and by our assumption proposition follows. Thus it remains to prove the claim.

Let $z_0 \in \Omega$ be such that

$$g(z_0) = \pm\ln\left(\sqrt{n}+\sqrt{n-1}\right) + \frac{i}{2}m\pi.$$

Now,

$$\begin{aligned}2\cosh\big(2g(z_0)\big) &= e^{2g(z_0)} + e^{-2g(z_0)}\\ &= e^{im\pi}\left(\left(\sqrt{n}+\sqrt{n-1}\right)^2 + \left(\sqrt{n}-\sqrt{n-1}\right)^2\right)\\ &= (-1)^m 2(2n-1)\\ \cosh\big(2g(z_0)\big) &= (-1)^m(2n-1).\end{aligned}$$

Then,

$$\begin{aligned}f(z_0) &= -\exp\big(\pi i\cosh\big(2g(z)\big)\big)\\ &= -\exp\big((2n-1)\pi i(-1)^m\big)\\ &= 1\end{aligned}$$

which is a contradiction. □

What we have proved is, if f is holomorphic on Ω which omits 0 and 1, then there exists a holomorphic function g on Ω which does not have a disk of radius 1 contained in $g(\Omega)$.

THEOREM 3 (Little Picard's Theorem). *If f is an entire function which omits two points, then f is a constant function.*

PROOF. Let z_0, z_1 be two distinct points which f omits. We may assume without loss of generality that $z_0 = 0$ and $z_1 = 1$,

$$f_1(z) = \frac{f(z)-z_0}{z_1-z_0}.$$

That is, f_1 is an entire function omits 0 and 1.

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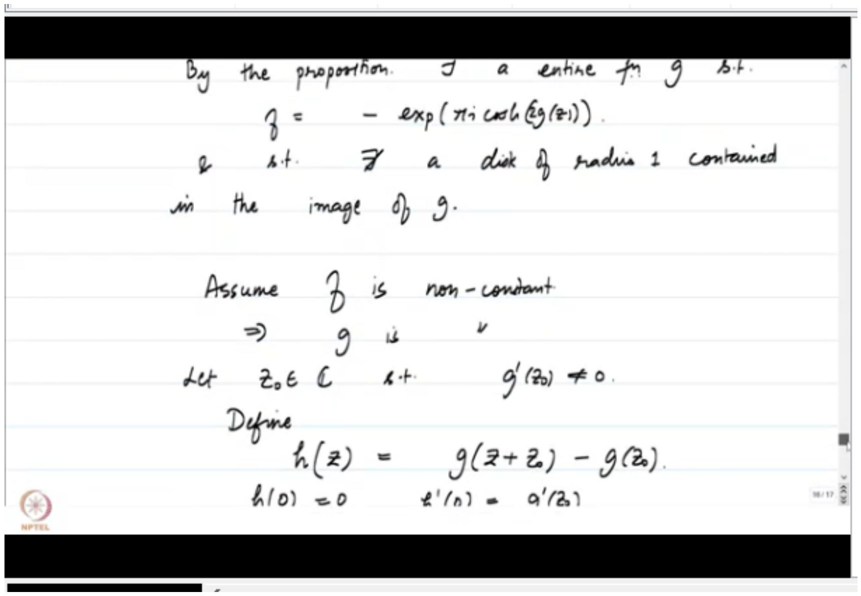

By the Proposition 1 and Proposition 2, there exists an entire function g such that

$$f(z) = -\exp\left(\pi i \cosh\left(2g(z)\right)\right)$$

which does not have a disk of radius 1 in the image of g.

Assume f is non-constant, then g is also non-constant.

Let $z_0 \in \mathbb{C}$ be such that $g'(z_0) \neq 0$. Define $h(z) = g(z+z_0) - g(z_0)$. Then $h(0) = 0$ and $h'(0) = g'(z_0)$. Now, it is left as an exercise to check that h is an entire function which also does not contain a disk of radius 1 in its image. Let $R > 0$ be some positive number and for $0 < r < R$, define

$$\psi(z) = \frac{1}{r} h\left(\frac{rz}{h'(0)}\right).$$

Then $\psi(0) = 0$ and $\psi'(0) = \frac{1}{r} h'(0) \frac{r}{h'(0)} = 1$.

By Bloch's theorem,

$$D\left(0, \frac{1}{72}\right) \subseteq \psi(\mathbb{D}).$$

Thus, for $w \in D\left(0, \frac{r}{72}\right)$, there exists $z \in \mathbb{D}$ such that

$$\frac{w}{r} = \frac{1}{r} h\left(\frac{rz}{h'(0)}\right) \implies w = h(z') \qquad \text{where } z' \in \mathbb{C}$$

$$\implies D\left(0, \frac{r}{72}\right) \subseteq h(\mathbb{C}).$$

But since r was arbitrary, we can choose $r > 72$, which is a contradiction.

Hence f is a constant function. $\square$