Complex Analysis

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Lecture No – 47

Bloch's Theorem

The theorem due to Bloch is a prerequisite for proving the 'Little Picard's theorem'. Bloch's theorem tries to answer the following question: suppose we have an open connected set Ω which contains $\overline{\mathbb{D}}$ and consider the family

$$\mathscr{F} = \left\{ f : \Omega \longrightarrow \mathbb{C} : f \text{ is holomorphic on } \Omega, f(0) = 0 \text{ and } f'(0) = 1 \right\}.$$

Then what is the largest disk that can be fitted up in $f(\mathbb{D})$ for any $f \in \mathscr{F}$.

To begin with, let us try to prove a special case of Bloch's theorem.

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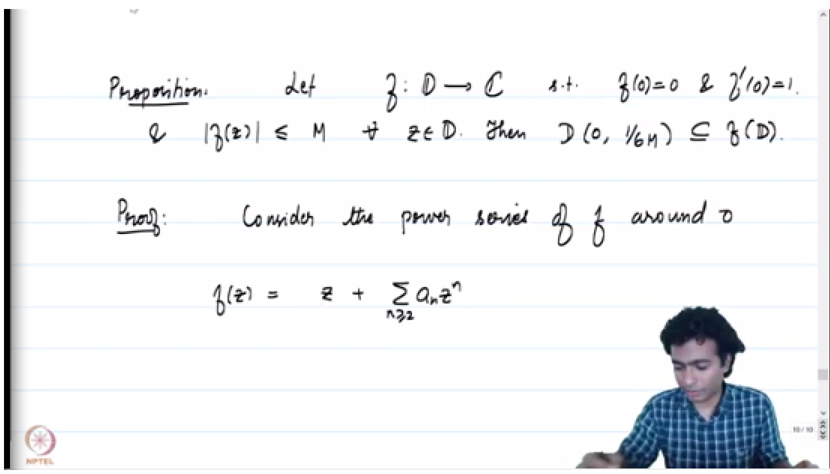

PROPOSITION 1. *Let $f : \mathbb{D} \longrightarrow \mathbb{C}$ be such that $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \leq M$ for each $z \in \mathbb{D}$. Then $D\left(0, \frac{1}{6M}\right) \subseteq f(\mathbb{D})$.*

PROOF. Consider the power series of f around 0,

$$f(z) = z + \sum_{n \geq 2} a_n z^n.$$

Note that
$$\left|f(z)\right| \geq |z| - \left|\sum_{n\geq 2} a_n z^n\right|.$$
By Cauchy estimates, we have
$$|a_n| \leq \frac{\sup\limits_{z\in\partial D(0,r)} \left|f(z)\right|}{r^n} \leq \frac{M}{r^n} \qquad \text{for each } 0<r<1.$$
Taking the limit as $r \longrightarrow 1^-$, we have $|a_n| \leq M$. Then,
$$\left|\sum_{n\geq 2} a_n z^n\right| \leq M\sum_{n\geq 2}|z|^n = M\frac{|z|^2}{1-|z|}.$$
Hence,
$$\left|f(z)\right| \geq |z| - \left(M\frac{|z|^2}{1-|z|}\right).$$
Our first observation is the number M should necessarily be at least 1. For proving this, let us first assume $M<1$, to arrive at a contradiction.

If $M<1$, then $f:\mathbb{D}\longrightarrow\mathbb{D}$ with $f(0)=0$ and $f'(0)=1$, then by Schwarz's lemma tells that $f(z)=\lambda z$ for $|\lambda|=1$. Then for $z\in\mathbb{D}$ such that $M<|z|<1$, we have $|f(z)|=|z|>M$, which is a contradiction. Hence $M\geq 1$.

Now, for $|z| = \frac{1}{4M} < 1$,
$$\begin{aligned}
|z| - M\frac{|z|^2}{1-|z|} &= \frac{1}{4M} - \frac{M\cdot\frac{1}{16M^2}}{1-\frac{1}{4M}} \\
&= \frac{1}{4M} - \frac{1}{12M} \\
&= \frac{2}{12M} \\
&= \frac{1}{6M}.
\end{aligned}$$
Hence $\left|f(z)\right| \geq \frac{1}{6M}$ on $\left\{z : |z| = \frac{1}{4M}\right\}$.

Let $w\in D\left(0,\frac{1}{6M}\right)$. Then, $|w| < \frac{1}{6M} \leq |f(z)|$ on $\left\{z : |z| = \frac{1}{4M}\right\}$. By Rouché's theorem, $f(z)-w$ and $f(z)$ has the same number of zeroes in $D\left(0,\frac{1}{4M}\right)$. Then there exists $z\in D\left(0,\frac{1}{4M}\right)$ such that $f(z)=w$. Hence $D\left(0,\frac{1}{6M}\right)\subseteq f(\mathbb{D})$. $\square$

Let us now generalize this result to bit higher generality by stating and proving the following proposition.

PROPOSITION 2. *Let $R > 0$, $f : D(0,R) \longrightarrow \mathbb{C}$ be a holomorphic function such that $f(0) = 0, |f'(0)| = \mu > 0$ and $|f(z)| \leq M$ for each $z \in D(0,R)$. Then*

$$D\left(0, \frac{R^2\mu^2}{6M}\right) \subseteq f(D(0,R)).$$

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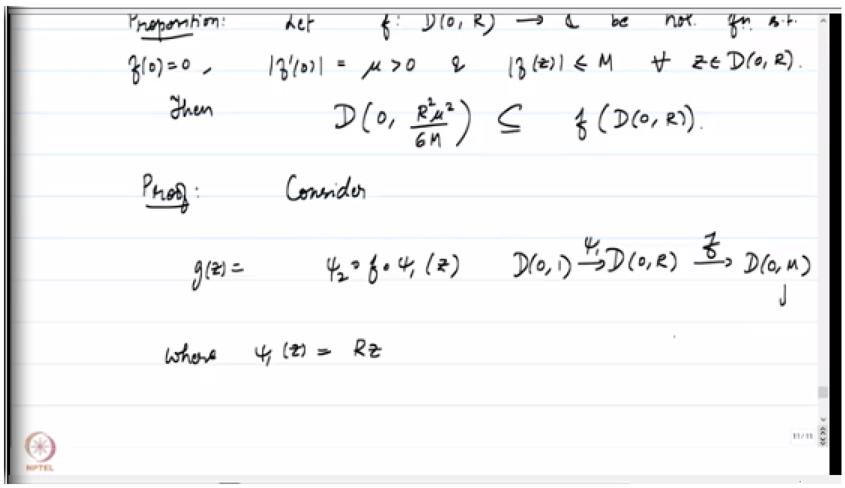

PROOF. Consider the function g on $\mathbb{D}$ given by,

$$g(z) = \psi_2 \circ f \circ \psi_1(z),$$

where $\psi_1(z) = Rz$ and $\psi_2(z) = \frac{z}{R\mu}$. Then note that

$$g : D(0,1) \xrightarrow{\psi_1} D(0,R) \xrightarrow{f} D(0,M) \xrightarrow{\psi_2} D\left(0, \frac{M}{R\mu}\right).$$

That is g is a map into $D\left(0, \frac{M}{R\mu}\right)$. Now, $g(0) = 0, g'(0) = 1$ and $|g(z)| \leq \frac{M}{R\mu}$ for $z \in \mathbb{D}$. Then by Proposition 1, we have $D\left(0, \frac{R\mu}{6M}\right) \subseteq g(\mathbb{D})$. Now it is left as an exercise to the reader to conclude the result. $\square$

Let us know prove the Bloch's theorem which is in slightly more generality.

THEOREM 3 (Bloch's theorem). *Let Ω be an open connected set in $\mathbb{C}$ such that $\overline{\mathbb{D}} \subset \Omega$. Suppose $f : \Omega \longrightarrow \mathbb{C}$ be such that $f(0) = 0$ and $f'(0) = 1$. Then there exists a disk D_1 contained in $\mathbb{D}$ such that $f\restriction_{D_1}$ is injective and*

$$D\left(0, \frac{1}{72}\right) \subseteq f(D_1) \subseteq f(\mathbb{D}).$$

PROOF. Define $M(r) = \sup\limits_{|z|=r} \left|f'(z)\right|$. Then $M(r)$ is continuous on $[0,1]$ (Why?). Now, define $h(r) := (1-r)M(r)$. Hence h is continuous on $[0,1]$. Moreover, we have $h(0) = 1$ and $h(1) = 0$.

Let $r_0 = \sup\{r : h(r) = 1\}$. Then $h(r) < 1$ for $r > r_0$. Since $M(r_0) = \sup\limits_{|z|=r_0} \left|f'(z)\right|$ and f is also continuous on the compact set $\{z : |z| = r_0\}$, there exists z_0 such that $|z_0| = r_0$ with $\left|f'(z_0)\right| = M(r_0)$. Now, $h(r_0) = 1 \implies \left|f'(z_0)\right| = \frac{1}{1-r_0}$.

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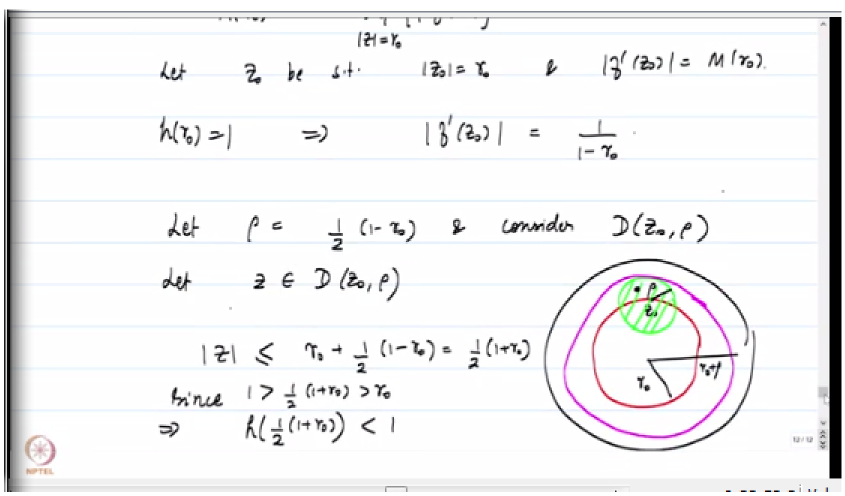

Let $\rho = \frac{1}{2}(1-r_0)$ and consider $D(z_0, \rho)$. Let $z \in D(z_0, \rho)$. Then,

$$|z| \leq r_0 + \frac{1}{2}(1-r_0) = \frac{1}{2}(1+r_0).$$

Since $1 > \frac{1}{2}(1+r_0) > r_0 \implies h\left(\frac{1}{2}(1+r_0)\right) < 1$. That is,

$$1 > \left(1 - \frac{1}{2}(1+r_0)\right) M\left(\frac{1}{2}(1+r_0)\right) \implies M\left(\frac{1}{2}(1+r_0)\right) < \frac{1}{\frac{1}{2}(1-r_0)} = \frac{1}{\rho}.$$

By maximum modulus principle,

$$\left|f'(z)\right| < \frac{1}{\rho} \qquad \text{for } z \in D(z_0, \rho).$$

Define $g(z) = f'(z) - f'(z_0)$ on $D(z_0, \rho)$. Then,

$$|g(z)| \le \left|f'(z)\right| + \left|f'(z_0)\right| < \frac{1}{\rho} + \frac{1}{2\rho} = \frac{3}{2\rho}.$$

Consider the holomorphic function given by $h(z) = \psi_2 \circ g \circ \psi_1(z)$ that is

$$h : \mathbb{D} \xrightarrow{\psi_1} D(z_0, \rho) \xrightarrow{f} D\left(0, \frac{3}{2\rho}\right) \xrightarrow{\psi_2} \mathbb{D},$$

where $\psi_1(z) = z_0 + \rho z$ and $\psi_2(z) = \dfrac{2\rho}{3} z$. Then $h(0) = 0$ and by Schwarz's lemma, we have

$$\left|\psi_2 \circ g \circ \psi_1(z)\right| \le |z| \implies \left|\frac{2\rho}{3} g(z)\right| \le \left|\psi_1^{-1}(z)\right| = \frac{|z - z_0|}{\rho}$$

$$\implies |g(z)| \le \frac{3}{2\rho^2} |z - z_0|.$$

Note that

$$\left|f'(z_0) - f'(z)\right| = |g(z)| < \frac{3}{\rho^2} |z - z_0| < \frac{3}{\rho^2} \cdot \frac{\rho}{3} = \frac{1}{2\rho} = |f'(z_0)|.$$

Put $D_1 := D\left(z_0, \dfrac{\rho}{3}\right)$. Let $z_1, z_2 \in D_1$. Then,

$$\begin{aligned}
\left|f(z_2) - f(z_1)\right| &= \left|\int_{\gamma_{z_1 \to z_2}} f'(z)\, dz\right| \\
&\ge \left|\int_{\gamma_{z_1 \to z_2}} f'(z_0)\, dz\right| - \left|\int_{\gamma_{z_1 \to z_2}} \left(f'(z_0) - f'(z)\right) dz\right| \\
&\ge \left|f'(z_0)\right| |z_1 - z_2| - \left|\int_{\gamma_{z_1 \to z_2}} \left(f'(z_0) - f'(z)\right) dz\right| \\
&> \left|f'(z_0)\right| |z_1 - z_2| - \left|f'(z_0)\right| |z_1 - z_2| = 0.
\end{aligned}$$

That is, for each $z_1, z_2 \in D_1$, we have $|f(z_2) - f(z_1)| > 0$. Hence f is injective on D_1.

Thus we proved one part of the Bloch's theorem. Let us now try to prove the remaining part of the theorem, which says that $D\left(0, \frac{1}{72}\right) \subseteq f(D_1)$.

Define $\tilde{g}(z) = f(z+z_0) - f(z_0)$ on $D\left(0, \frac{\rho}{3}\right)$. Then, we have $\tilde{g}(0) = 0$ and $\left|\tilde{g}'(0)\right| = \left|f'(z_0)\right| = \frac{1}{2\rho}$.

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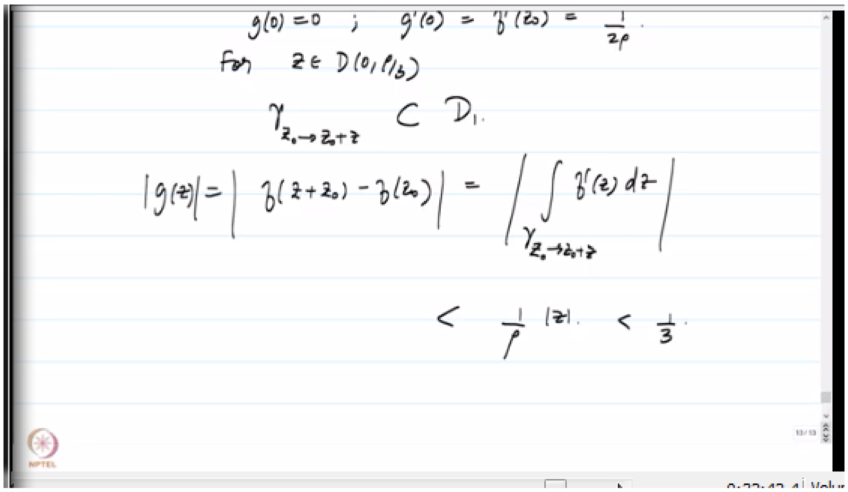

For $w \in D\left(0, \frac{\rho}{3}\right)$, $\gamma_{z_0 \to z_0+w} \subset D_1$. Then

$$\left|\tilde{g}'(w)\right| = \left|f(w+z_0) - f(z_0)\right| = \left|\int_{\gamma_{z_0 \to z_0+w}} f'(z)\,dz\right| < \frac{1}{\rho}|w| < \frac{1}{3}.$$

Hence $\tilde{g} : D\left(0, \frac{\rho}{3}\right) \longrightarrow \mathbb{C}$ is such that $\tilde{g}(0) = 0$, $|\tilde{g}(0)| = \dfrac{1}{2\rho}$ and $|\tilde{g}| < \dfrac{1}{3}$ for $z \in \mathbb{D}$. Then by Proposition 1, we have

$$D\left(0, \frac{\left(\frac{\rho}{3}\right)^2 \left(\frac{1}{2\rho}\right)^2}{6 \cdot \frac{1}{3}}\right) \subseteq \tilde{g}\left(D\left(0, \frac{\rho}{3}\right)\right)$$

$$\implies D\left(0, \frac{1}{72}\right) \subseteq \tilde{g}\left(D\left(0, \frac{\rho}{3}\right)\right)$$

$$\implies D\left(0, \frac{1}{72}\right) \subseteq f(D_1).$$

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