Complex Analysis

Prof. Pranav Haridas

Kerala School of Mathematics

Lecture No – 46

Covering Spaces

We defined the notion of a lift of a map g with respect to a continuous map f. We also discussed some of the properties of such a lift and we saw that if we have a couple of curves γ_1, γ_2 which are homotopic to each other and at every stage γ_s could be lifted, then the homotopy could also be lifted. However, at no point of time, we did talk about the existence of such a lift. So let us define a 'covering map' which gives us a sufficient condition to talk about when a curve could be lifted.

DEFINITION 1 (Covering map). Let X, Y be open subsets of $\mathbb{C}$. We say that a continuous map $f : Y \longrightarrow X$ is a covering map if given $x \in X$, there exists a neighborhood U of x and open sets $\{V_\alpha\}_{\alpha \in A}$ in Y such that $f^{-1}(U) = \coprod_{\alpha \in A} V_\alpha$ and $f \restriction_{V_\alpha} : V_\alpha \longrightarrow U$ is a homeomorphism. Then Y is called a cover of X. The set U is called as an evenly covered neighborhood.

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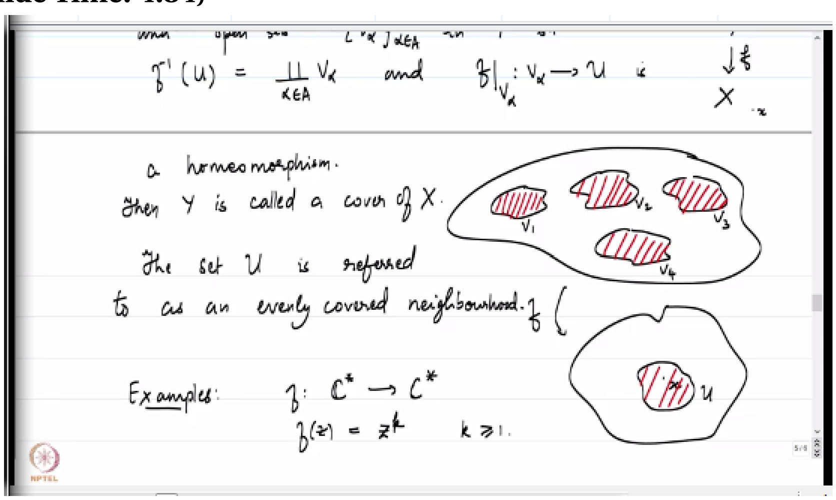

EXAMPLE 1.

- Let $f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ be given by $f(z) = z^k$ for $k \in \mathbb{N}$. Since $f'(z) \neq 0$ for each $z \in \mathbb{C}^*$, we have f is a local biholomorphism. Given $z_0 \in \mathbb{C}^*, \exists\, w_1, \dots, w_k \in \mathbb{C}^*$ such that $z^k = z_0$ has w_i as solutions. Let V_i be neighborhoods of w_i and U_i be neighborhood of z_0 such that $f\restriction_{V_i}: V_i \longrightarrow U_i$ is a homeomorphism. Define $U = \bigcap_{i=1}^{n} U_i$ and $V_i' = f^{-1}(U) \cap V_i$. Then, $f^{-1}(U) = \coprod_{i=1}^{n} V_i'$. Hence f is a covering map.
- Consider $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$. Since exp is a local biholomorphism, given $z_0 \in \mathbb{C}^*$, there exists $w_0 \in \mathbb{C}$ such that $e^{w_0} = z_0$. By inverse function theorem, we have neighborhoods V and U of w_0 and z_0 respectively such that $\exp\restriction_V V :\longrightarrow U$ is a biholomorphism. Now, the reader should verify that if

$$V_n := V + 2\pi i n = \{z + 2\pi i n : z \in V, n \in \mathbb{Z}\},$$

then $\{V_n\}_{n\in\mathbb{Z}}$ is mutually disjoint. Then $\exp\restriction_{V_n}: V_n \longrightarrow U$ is a homeomorphism and $\exp^{-1}(U) = \coprod_{n\in\mathbb{Z}} V_n$. Hence exp is a covering map

THEOREM 2. *Let $f : Y \longrightarrow X$ be a covering map and $\gamma : [a,b] \longrightarrow X$ be a curve from x_0 to x_1 in X. Suppose $y_0 \in f^{-1}(\{x_0\})$. Then there exists unique lift $\tilde{\gamma} : [a,b] \longrightarrow Y$ of γ with respect to f such that $\tilde{\gamma}(a) = y_0$.*

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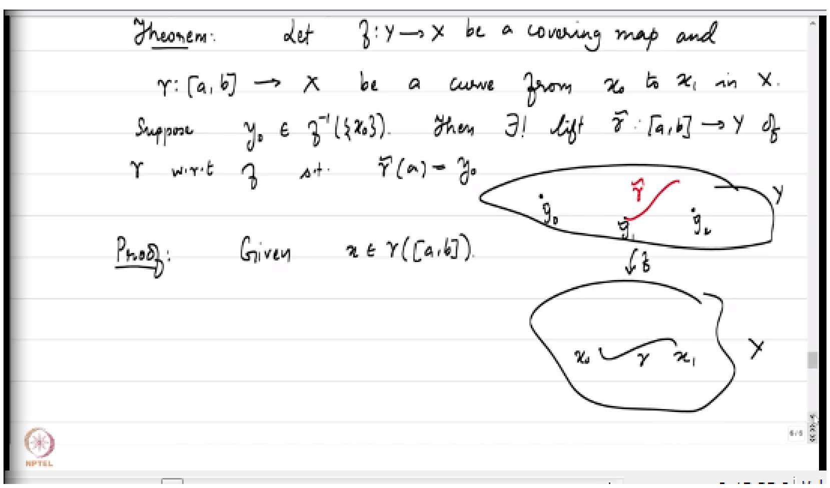

Given below is a corrected version of the proof given in the lecture.

PROOF. Given $x \in \gamma([a,b])$. We have a neighborhood U_x evenly covered for the covering map f.

Let $\mathscr{U} := \{U_x : x \in \gamma([a,b])\}$ be the collection of evenly covered neighborhoods corresponding to each point in the image of γ. Then $\mathscr{U}$ is an open cover for $\gamma([a,b])$. Since γ is continuous, $\mathscr{U}' = \{\gamma^{-1}(U_x) : U_x \in \mathscr{U}\}$ is an open cover for the compact set $[0,1]$. Let $\delta > 0$ be the Lebesgue number of $\mathscr{U}'$.

Consider a partition $P : a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$ with partition size less than $\frac{\delta}{2}$, so that image of the consecutive points under γ lie in one evenly covered neighborhood.

We will construct a lift by induction on the index t_i. For t_0, t_1, there exists an evenly covered neighborhood $U_1 \in \mathscr{U}$ such that $\gamma(t_0), \gamma(t_1) \in U_1$. Let $f^{-1}(U_1) = \coprod_{\alpha \in A} V_\alpha$.

Let $\alpha_0 \in A$ be such that $y_0 \in V_{\alpha_0}$ and $\varphi : U_1 \longrightarrow V_{\alpha_0}$ be the inverse of $f \restriction_{V_{\alpha_0}}$. For $t \in [t_0, t_1]$, define

$$\tilde{\gamma}(t) = \varphi \circ \gamma(t).$$

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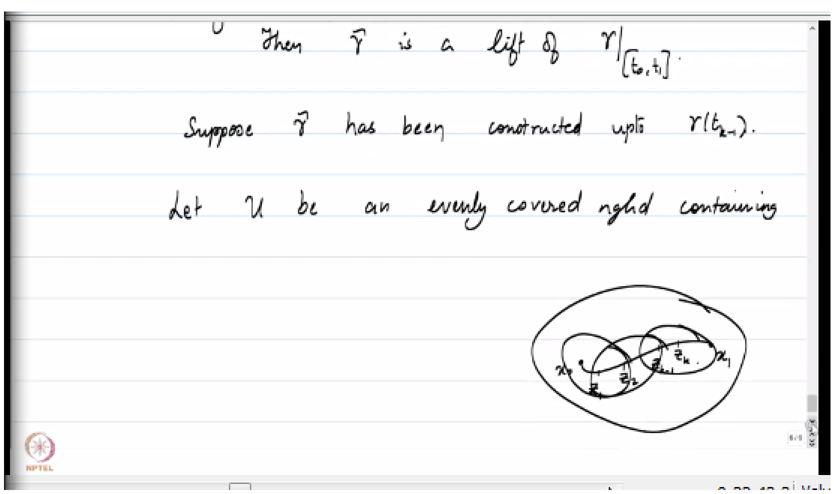

Then $\tilde{\gamma}$ is a lift of $\gamma \restriction_{[t_0,t_1]}$. Suppose $\tilde{\gamma}$ has been constructed inductively up to $\gamma(t_{k-1})$. Let $U_k \in \mathscr{U}$ be an evenly covered neighborhood containing $\gamma(t_{k-1})$ and $\gamma(t_k)$. Let us put $\tilde{\gamma}(t_{k-1}) = \tilde{z}_{k-1}$. Suppose $f^{-1}(U_k) = \coprod_{\alpha' \in A'} V'_\alpha$ and $\alpha'_0 \in A'$ be such that $\tilde{z}_{k-1} \in V'_{\alpha'_0}$. Let

$\varphi' : V'_{\alpha'_0} \longrightarrow U_k$ be the inverse of $f\restriction_{V'_{\alpha'_0}}$. For $t \in [t_{k-1}, t_k]$, define

$$\tilde{\gamma}(t) = \varphi' \circ \gamma(t).$$

Also $\tilde{\gamma} : [a, t_k] \longrightarrow Y$ is a lift of $\gamma\restriction_{[a,t_k]}$. Hence, inductively, we have $\tilde{\gamma} : [a, b] \longrightarrow Y$ is a lift of γ. □

EXAMPLE 3. Consider the function $f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ given by $f(z) = z^2$. Let $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. For $t \in [0, 2\pi]$ $\tilde{\gamma}(t) = e^{i\frac{t}{2}}$. Then, note that $f\left(\tilde{\gamma}(t)\right) = e^{it} = \gamma(t)$. Hence $\tilde{\gamma}$ is a lift of γ with respect to f with initial point 1.

PROPOSITION 4. *Let X be path connected and $f : Y \longrightarrow X$ be a covering map. Suppose $x_0, x_1 \in X$. Then the cardinality of $f^{-1}(x_0)$ is the same as the cardinality of $f^{-1}(x_1)$.*

PROOF. Since X is path connected, there exists a path $\gamma : [a, b] \longrightarrow X$ from x_0 to x_1. Let $y_0 \in f^{-1}(x_0)$. Then, by Theorem 2, there exists a lift $\tilde{\gamma} : [a, b] \longrightarrow Y$ of γ with initial point y_0.

Define $\varphi : f^{-1}(x_0) \longrightarrow f^{-1}(x_1)$ given by $\varphi(y) = \tilde{\gamma}(b)$. By uniqueness, this map is injective. Now, it is left as an exercise for the reader to prove that φ is in fact a bijection. □

THEOREM 5. *Let X, Y be open subsets of $\mathbb{C}$ and $f : Y \longrightarrow X$ be a covering map from Y to X. Let Z be an open connected subset of $\mathbb{C}$, which is simply connected and locally connected. Suppose $g : Z \longrightarrow X$ be a continuous function. If given $z_0 \in Z$ and $y_0 \in Y$ such that $g(z_0) = f(y_0)$, then there exists a unique lift $\tilde{g} : Z \longrightarrow Y$ of g with respect to f.*

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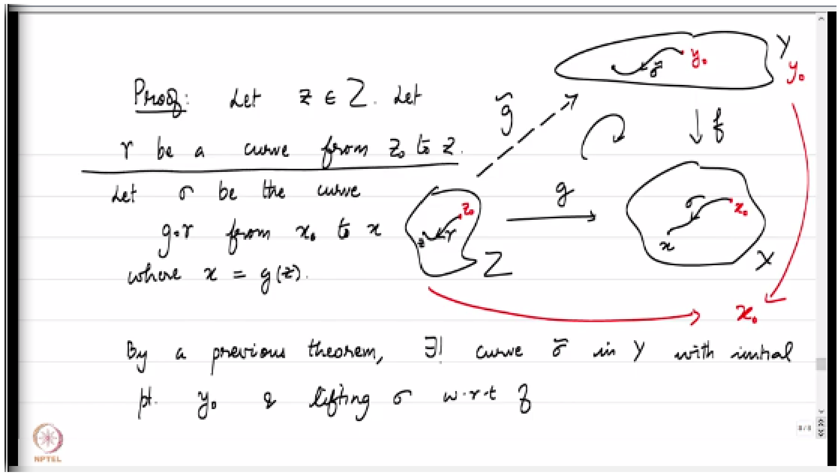

PROOF. Since every connected open subsets of $\mathbb{C}$ are path connected, we have Z is path connected. Let $z \in Z$ and let $\gamma_0 : [a,b] \longrightarrow Z$ be a curve from z_0 to z. Let $\sigma_0 : [a,b] \longrightarrow X$ be the curve $g \circ \gamma_0$ from $g(z_0) = x_0$ to $g(z) = x$.

By Theorem 2, there exists a unique curve $\tilde{\sigma}_0 : [a,b] \longrightarrow Y$ in Y with initial point y_0 and lifting σ_0 with respect to f. Define $\tilde{g}(z) := \tilde{\sigma}_0(b)$.

Let us try to prove that $\tilde{g}$ is well defined. Let γ_1 be another curve in Z from z_0 to z. Since Z is simply connected, γ_0 is homotopic to γ_1 with fixed end points through the homotopy $H : [0,1] \times [a,b] \longrightarrow Z$. Now one can easily verify that $g \circ H$ is a homotopy with fixed end points from σ_0 to $\sigma_1 = g \circ \gamma_1$.

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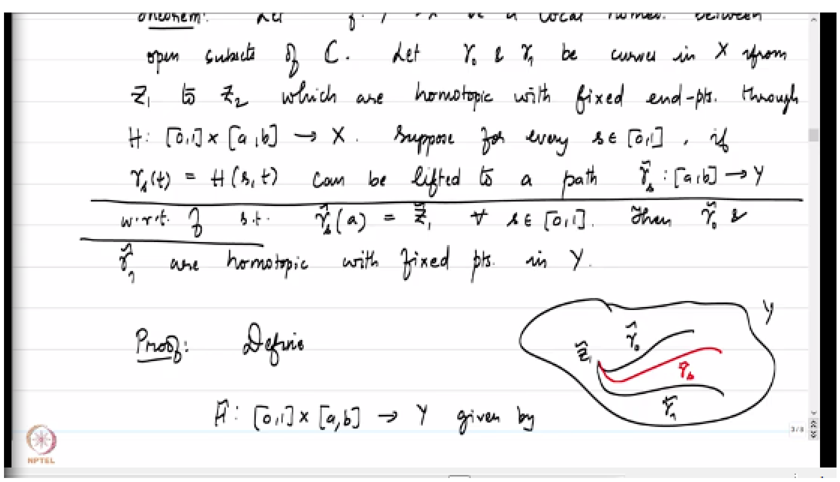

If $\sigma_s(t) := g \circ H(s,t)$ is a curve from x_0 to x, then there exists a lift $\tilde{\sigma}_s$ in Y of σ_s with initial point y_0 (by Theorem 2). Also, $g \circ H$ can be lifted to a homotopy with fixed end points in Y from $\tilde{\sigma}_0$ to $\tilde{\sigma}_1$. In particular, $\tilde{\sigma}_1(b) = \tilde{g}(z)$. Hence $\tilde{g}$ is well defined.

Now it is an easy check that $f\left(\tilde{g}(z)\right) = g(z)$.

Claim: $\tilde{g}$ is continuous.

To prove that $\tilde{g}$ is continuous, it is enough to show that, given $z \in Z$, let V be a neighborhood of $\tilde{g}(z) = y$. Then z is an interior point of $\tilde{g}^{-1}(V)$.

By shrinking V, if needed, we may assume that $f\restriction_V : V \longrightarrow U$ is a homeomorphism with inverse $\varphi : U \longrightarrow V$, where U is an evenly covered neighborhood.

Since g is continuous, consider $g^{-1}(U)$ which is a neighborhood of of z. By local connectedness of Z, there exists a neighborhood W of z such that $W \subseteq g^{-1}(U)$ and W is connected.

Claim: $W \subseteq \tilde{g}^{-1}(V)$.

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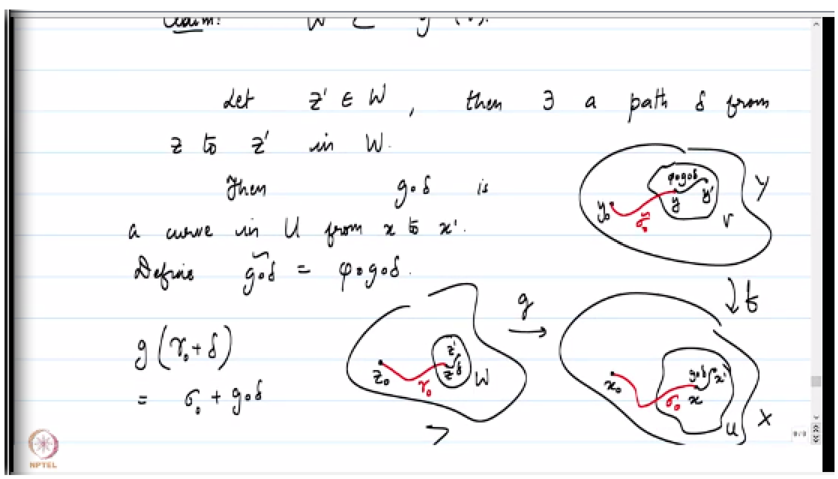

Let $z' \in W$, then there exists a path $\delta : [a,b] \longrightarrow W$ from z to z' in W. Then $g \circ \delta$ is a curve in U from x to $x' = g(z')$. Now we will lift the curve $g \circ \delta$ to a curve in V. Define $\widetilde{g \circ \delta} = \varphi \circ g \circ \delta$.

A simple check will tell us that $g(\gamma_0 + \delta) = g \circ \gamma_0 + g \circ \delta = \sigma_0 + g \circ \delta$ and further we have $\tilde{\sigma}_0 + \widetilde{g \circ \delta} = \tilde{\sigma}_0 + \varphi \circ g \circ \delta$ is a lift of $\sigma_0 + g \circ \delta$.

By the definition of $\tilde{g}$,

$$\tilde{g}(z') = \left(\tilde{\sigma}_0 + \varphi \circ g \circ \delta\right)(b) = \varphi \circ g \circ \delta(b) \in V.$$

Hence $z' \in \tilde{g}^{-1}(V) \implies W \subseteq \tilde{g}^{-1}(V)$. $\square$

COROLLARY 6. *Let* $\Omega \subseteq \mathbb{C}$ *be an open connected subset which is simply connected and locally connected and* $g : \Omega \longrightarrow \mathbb{C}^*$ *be a holomorphic map. Then there exists a lift* $\tilde{g} : \Omega \longrightarrow \mathbb{C}$ *such that* $\exp(\tilde{g}) = g$.