Complex Analysis

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Lecture No – 45

Lifting of maps

Picard's theorem is one of the classical and powerful results in complex analysis. Very broadly, Picard's theorem tells us that if you consider a non-constant entire function, then its image does not miss too many points. We will make it concrete and prove Picard's theorem in the due course. However, before that let us develop some machinery that is needed to indeed prove Picard's theorem namely that of 'covering spaces'.

First, we will define what is meant by the lifting property of maps and we will prove some properties of lifts.

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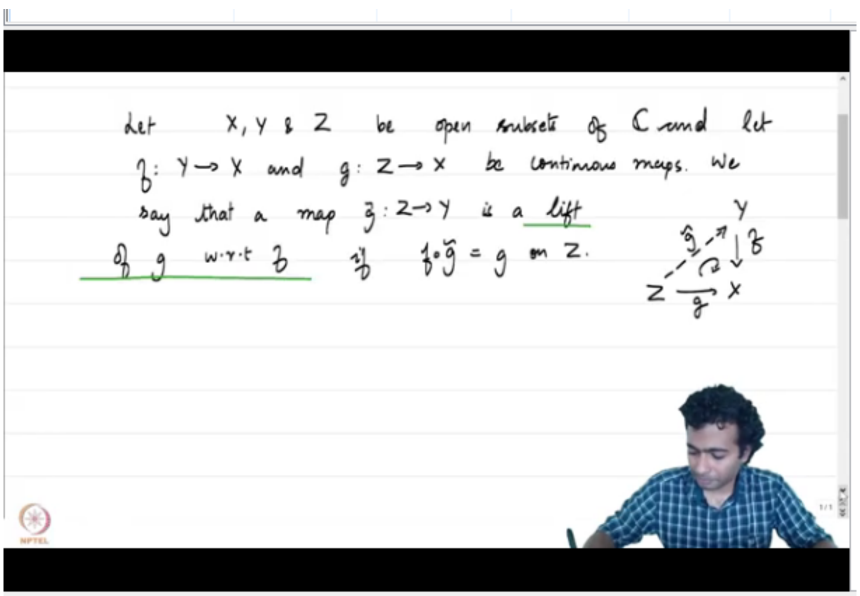

DEFINITION 1. Let X, Y and Z be open subsets of $\mathbb{C}$ and let $f : Y \longrightarrow X$ and $g : Z \longrightarrow X$ be continuous maps. We say that a map $\tilde{g} : Z \longrightarrow Y$ is a lift of g with respect to f if $f \circ \tilde{g} = g$ on Z.

EXAMPLE 1. Consider the exponential function $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$ and let Ω be a simply connected subset of $\mathbb{C}^*$. Then let $\tilde{id}$ be a branch of the logarithm holomorphic on Ω. That is, $\exp\left(\widetilde{Id}(z)\right) = z = Id(z)$. Hence a continuous branch of the logarithm is a lift of the identity map with respect to the exponential map.

THEOREM 2 (Uniqueness of Lifts). *Let X, Y be open subsets of $\mathbb{C}$ and $f : Y \longrightarrow X$ be a local homeomorphism. (That is, given $y_0 \in Y$, there exists a neighborhood V of y_0 in Y and U of $f(y_0)$ in X such that $f\restriction_V : V \longrightarrow U$ is a homeomorphism onto U.) Let be Z be a connected open subset of $\mathbb{C}$ and $g : Z \longrightarrow X$ be a continuous map. Let $\tilde{g}_1$ and $\tilde{g}_2$ be lifts of g with respect to f and suppose $\tilde{g}_1(z_0) = \tilde{g}_2(z_0)$ for some $z_0 \in Z$. Then $\tilde{g}_1 \equiv \tilde{g}_2$.*

PROOF. Let $E = \left\{z \in Z : \tilde{g}_1(z) = \tilde{g}_2(z)\right\}$. Notice that $E \neq \emptyset$ as $z_0 \in E$ and also E is closed since $E = \left(\tilde{g}_1 - \tilde{g}_2\right)^{-1}(0)$.

Claim: E is open in Z.

Let $z \in E$. Then $\tilde{g}_1(z) = \tilde{g}_2(z) = y$ (say). Let $x = f(y)$. Since f is a local homeomorphism, there exists a neighborhood V of y and U of x in Y and X respectively such that $f\restriction_V : V \longrightarrow U$ is a homeomorphism. Let $\phi : U \longrightarrow V$ be the inverse of $f\restriction_V$.

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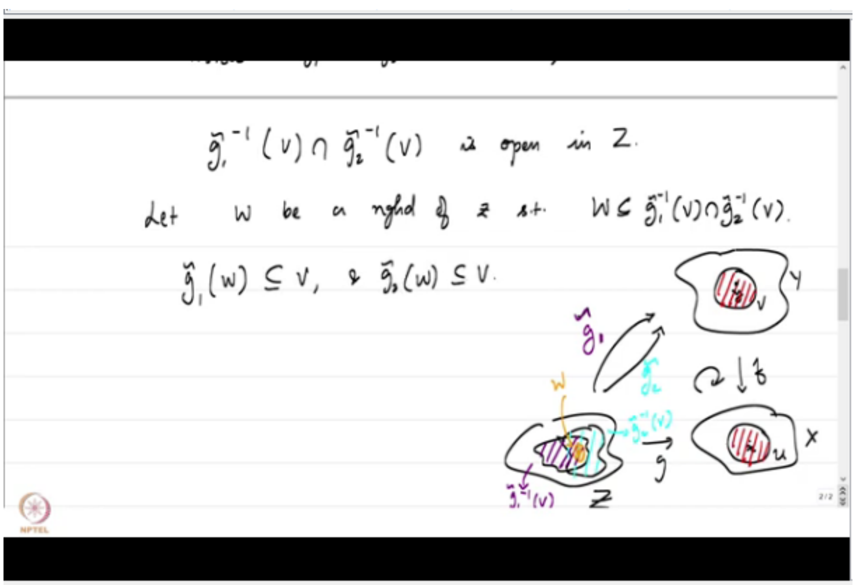

Since $\tilde{g}_1$ and $\tilde{g}_2$ are continuous, we have $\tilde{g}_1^{-1}(V) \cap \tilde{g}_2^{-1}(V)$ is open in Z and contains z. Let W be a neighborhood of z such that $W \subseteq \tilde{g}_1^{-1}(V) \cap \tilde{g}_2^{-1}(V)$. Observe that $\tilde{g}_1(W) \subseteq V$ and $\tilde{g}_2(W) \subseteq V$. Then, by the construction, we have

$$g \restriction_W = f \circ \tilde{g}_1 \restriction_W = f \restriction_V \circ \tilde{g}_1 \restriction_W$$

and similarly

$$g \restriction_W = f \circ \tilde{g}_2 \restriction_W = f \restriction_V \circ \tilde{g}_2 \restriction_W .$$

Then by composing with ϕ, we have

$$\tilde{g}_1 \restriction_W = \phi \circ g \restriction_W = \tilde{g}_2 \restriction_W .$$

Hence $\tilde{g}_1 = \tilde{g}_2$ on $W \implies W \subseteq E$. Therefore E is open in Z. Since Z is connected, $E = Z$. $\square$

Till now, we have not addressed the case of holomorphic maps, we have only dealt with continuous maps. Let us address that issue now.

THEOREM 3. *Let $f : Y \longrightarrow X$ be a holomorphic map such that $f'(y) \neq 0$ for each $y \in Y$. Let $g : Z \longrightarrow X$ be a holomorphic map such that $\tilde{g} : Z \longrightarrow Y$ is a lift of g with respect to f. Then $\tilde{g}$ is holomorphic.*

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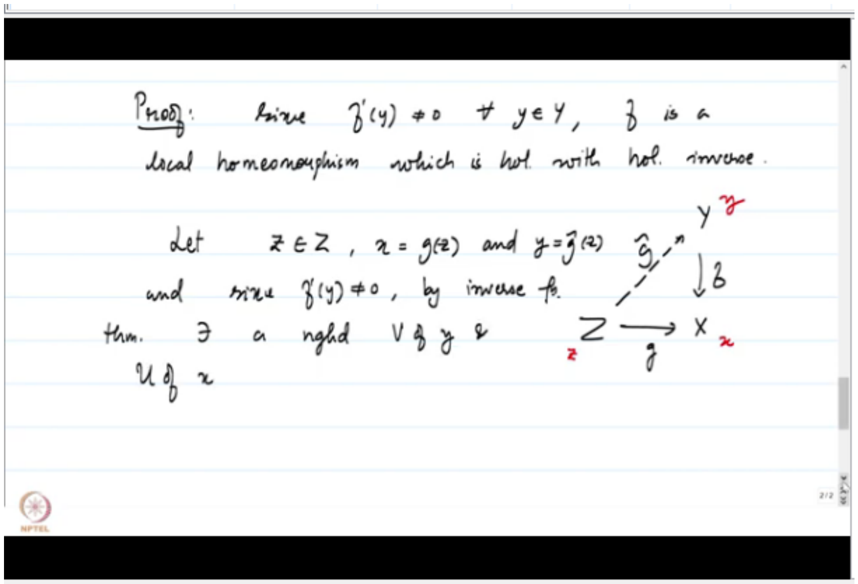

Proof. Since $f'(y) \neq 0$ for every $y \in Y$, f is a local homeomorphism which is holomorphic with holomorphic inverse.

Let $z \in Z$, $x = g(z)$ and $y = \tilde{g}(z)$. Since $f'(y) \neq 0$, by inverse function theorem, there exists a neighborhood V of y and U of x such that $f\restriction_V \longrightarrow U$ is a holomorphic map onto U with holomorphic inverse $\phi : U \longrightarrow V$.

Let $W = f^{-1}(V)$. On W, we have $\tilde{g} = \phi \circ g$. Since g and ϕ are holomorphic, we have $\tilde{g}$ is holomorphic on $W \implies \tilde{g}$ is holomorphic on Z. $\square$

Let us now try to see what happens when we lift curves. Curves are very special objects for us. And we would like to see what happens when we are lifting curves under such good maps like say local homeomorphisms. The uniqueness of the lifting tells us that if you have if you prescribe the initial point of the lifting then the lifting has to be unique of a curve. That is, let $f : Y \longrightarrow X$ be a local homeomorphism. Suppose $\gamma : [a,b] \longrightarrow X$ be a curve and $\tilde{\gamma} : [a,b] \longrightarrow Y$ be a lift of γ with respect to f. Then $\tilde{\gamma}$ is unique if we prescribe the initial point.

Theorem 4. *Let $f : Y \longrightarrow X$ be a local homeomorphism between open subsets of $\mathbb{C}$. Let γ_0 and γ_1 be curves in X from z_1 to z_2 which are homotopic with fixed end points* *through* $H : [0,1] \times [a,b] \longrightarrow X$*. Suppose for every* $s \in [0,1]$*, if* $\gamma_s(t) = H(s,t)$ *can be lifted to a path* $\tilde{\gamma}_s : [a,b] \longrightarrow Y$ *with respect to* f *such that* $\tilde{\gamma}_s(a) = \tilde{z}_1$ *for every* $s \in [0,1]$*. Then* $\tilde{\gamma}_0$ *and* $\tilde{\gamma}_1$ *are homotopic with fixed end points in* Y*.*

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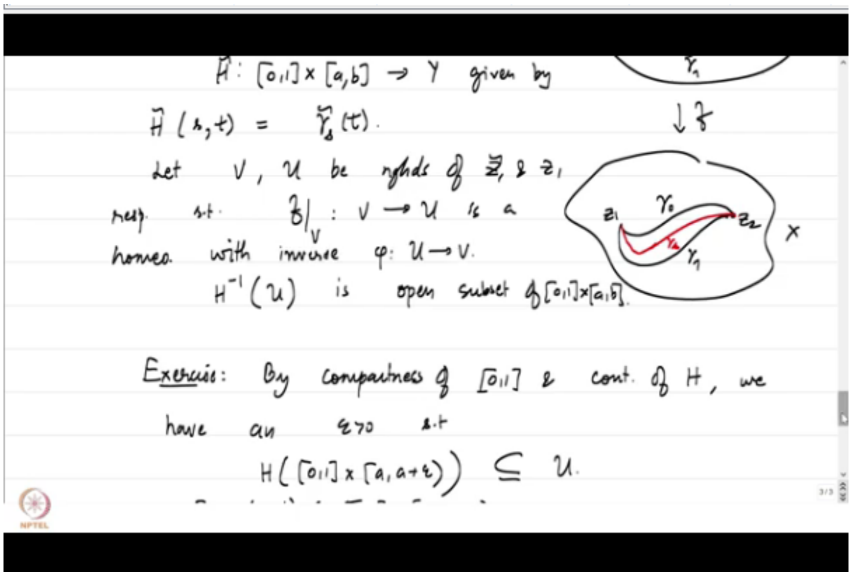

PROOF. Define $\widetilde{H} : [0,1] \times [a,b] \longrightarrow Y$ given by $\widetilde{H}(s,t) = \tilde{\gamma}_s(t)$. Let us try to prove this map is continuous, which is the important part.

Let V, U be neighborhoods of $\tilde{z}_1$ and z_1 respectively such that $f\restriction_V : V \longrightarrow U$ is a homeomorphism with inverse $\phi : U \longrightarrow V$. Since H is continuous, $H^{-1}(U)$ is an open subset of $[0,1] \times [a,b]$.

Check that, by the compactness of $[0,1]$ and continuity of H, we have an $\epsilon > 0$ such that $H([0,1] \times [a, a+\epsilon)) \subseteq U$.

For $(s,t) \in [0,1] \times [a, a+\epsilon)$, we have $f \circ \tilde{\gamma}_s(t) = \gamma_s(t)$. By composing with ϕ, we have

$$\widetilde{H}(s,t) = \phi \circ \gamma_s(t) = \phi \circ H(s,t).$$

Hence $H\restriction_{[0,1]\times[a,a+\epsilon)}$ is continuous. We will prove it is continuous on $[0,1] \times [a,b]$.

Suppose $\widetilde{H}$ is not continuous. Given $s_0 \in [0,1]$, let $t_0 \in [a,b]$ be such that $\widetilde{H}$ is not continuous on at (s_0, t_0) and $\widetilde{H}$ is continuous at (s_0, t) for each $t < t_0$.

Let U' and V' be neighborhoods of $\gamma_{s_0}(t_0)$ and $\tilde{\gamma}_{s_0}(t_0)$ such that $f\restriction_{V'}: V' \longrightarrow U'$ is a homeomorphism and $\phi' : U' \longrightarrow V'$ be its continuous inverse.

Since H is continuous, there exists $\delta > 0$ such that

$$H((s_0-\delta, s_0+\delta)\times(t_0-\delta, t_0+\delta)) \subseteq U'.$$

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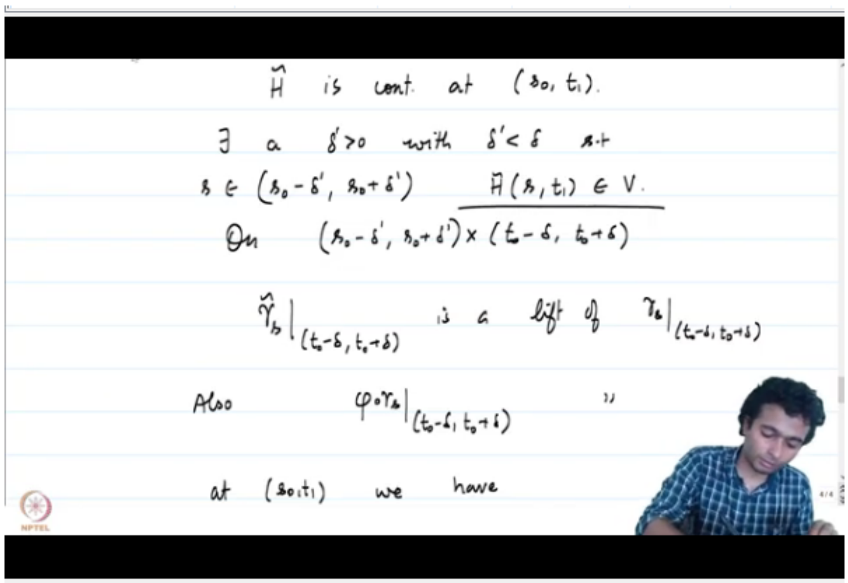

Let $t_1 \in (t_0-\delta, t_0)$. Then $\widetilde{H}$ is continuous at (s_0, t_1). Then there exists a $\delta' > 0$ with $\delta' < \delta$ such that for $s \in (s_0-\delta', s_0+\delta')$, we have $\widetilde{H}(s, t_1) \in V'$.

On $(s_0-\delta', s_0+\delta')\times(t_0-\delta, t_0+\delta)$, $\tilde{\gamma}_s\restriction_{(t_0-\delta,t_0+\delta)}$ is a lift of $\gamma_s\restriction_{(t_0-\delta,t_0+\delta)}$. Also note that $\phi\circ\gamma_s\restriction_{(t_0-\delta,t_0+\delta)}$ is a lift of $\gamma_s\restriction_{(t_0-\delta,t_0+\delta)}$.

Observe that, at (s_0, t_1), we have

$$\tilde{\gamma}_s(s_0, t_1) = \phi\circ\gamma_s(s_0, t_1)$$

$$\implies \tilde{\gamma}_s\restriction_{(t_0-\delta,t_0+\delta)} = \phi\circ\gamma_s\restriction_{(t_0-\delta,t_0+\delta)}$$

$$\implies \widetilde{H}(s,t) = \phi\circ H(s,t)$$

for $(s,t) \in (s_0-\delta', s_0+\delta')\times(t_0-\delta, t_0+\delta)$.

Since $\phi \circ H$ is continuous at (s_0, t_0), we have a contradiction. Hence $\widetilde{H}$ is continuous on $[0,1] \times [a,b]$.

Notice that $f \circ \widetilde{H}(s,b) = H(s,b) = z_1$. In particular, $\widetilde{H}(s,b) \in f^{-1}\{z_1\}$. Since $f^{-1}\{z_1\}$ is discrete in Y and $\widetilde{H}$ is continuous on $[0,1] \times \{b\}$, we have $\widetilde{H}([0,1] \times \{b\})$ is connected subset of Y contained in $f^{-1}\{z_1\}$. Hence $\widetilde{H}([0,1] \times \{b\})$ is a constant. $\square$