Complex Analysis

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Lecture No – 44

Problem Session

PROBLEM 1. Let Ω be a simply connected domain and $f : \Omega \longrightarrow \mathbb{C}^*$ be a function which is holomorphic on Ω. Then prove that there exists a function $h : \Omega \longrightarrow \mathbb{C}$ such that h is holomorphic and $(h(z))^n = f(z)$.

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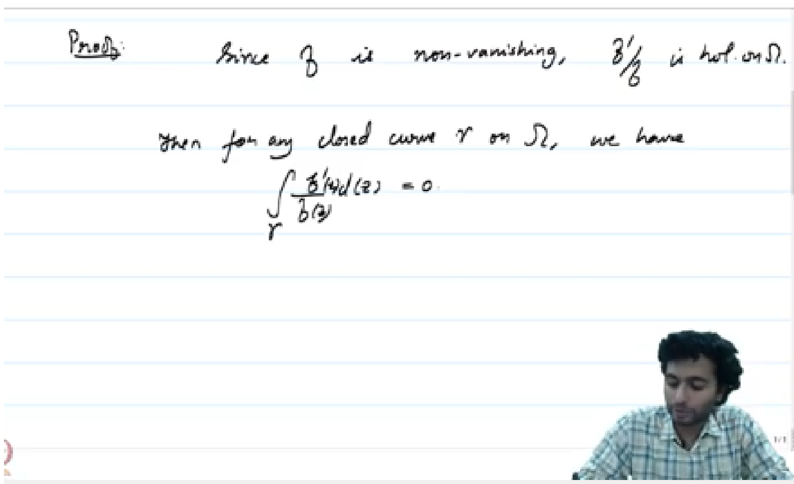

SOLUTION 1. Since f is non-vanishing, $\frac{f'}{f}$ is holomorphic on Ω. Then for any closed curve γ on Ω, we have

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0.$$

By the fundamental theorem of calculus, there exists an anti-derivative g of $\frac{f'}{f}$ on Ω.

Let $z_0 \in \Omega$ and $w_0 \in \mathbb{C}^*$ be such that $f(z_0) = w_0$ and g be picked in such a manner that $e^{g(z_0)} = w_0$ (This can be done as the anti-derivative is determined uniquely up to addition by a constant.).

Consider the function $\dfrac{\exp g(z)}{f(z)}$. Then,

$$\begin{aligned}\frac{d}{dz}\left(\frac{\exp g(z)}{f(z)}\right) &= \frac{f(z)g'(z)\exp(g(z)) - \exp(g(z))f'(z)}{\big(f(z)\big)^2}\\ &= \frac{f'(z)\exp(g(z)) - f'(z)\exp(g(z))}{\big(f(z)\big)^2}\\ &= 0.\end{aligned}$$

Therefore, $\dfrac{\exp g(z)}{f(z)} = \text{constant}$. For $z = z_0$, we have $\dfrac{\exp g(z_0)}{f(z_0)} = \dfrac{w_0}{w_0} = 1$.

Hence,

$$\frac{\exp g(z)}{f(z)} = 1 \implies \exp(g(z)) = f(z).$$

Now, let us define $h(z) = \exp\left(\dfrac{g(z)}{n}\right)$. Then,

$$(h(z))^n = f(z).$$

PROBLEM 2. Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic map with two fixed points. Then prove that f is identity.

SOLUTION 2. Let α and β be points in $\mathbb{D}$ such that $f(\alpha) = \alpha$ and $f(\beta) = \beta$. Define $g : \mathbb{D} \longrightarrow \mathbb{D}$, given by

$$g(z) = \varphi_\alpha \circ f \circ \varphi_{-\alpha}(z),$$

where $\varphi_\alpha(z) = \dfrac{z - \alpha}{1 - \bar{\alpha}z}$. Then $g(0) = 0$.

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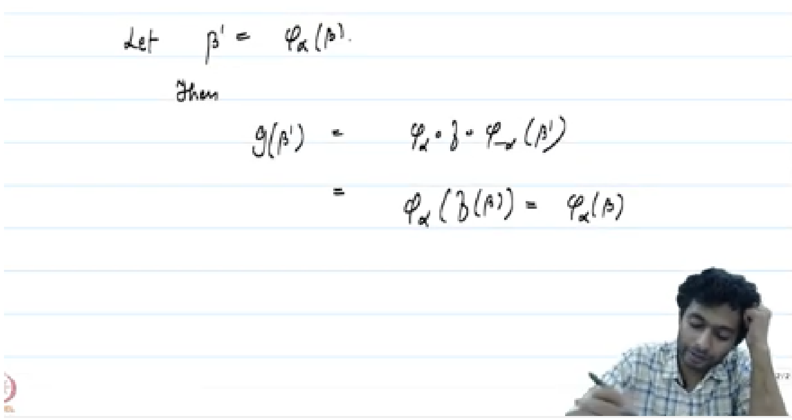

Let $\beta' = \varphi_\alpha(\beta)$. Then,

$$\begin{aligned} g(\beta') &= \varphi_\alpha \circ f \circ \varphi_{-\alpha}(\beta') \\ &= \varphi_\alpha\left(f(\beta)\right) \\ &= \varphi_\alpha(\beta) \\ &= \beta'. \end{aligned}$$

Hence by Schwarz's lemma, $g(z) = \lambda z$ where $|\lambda| = 1$. Since $g(\beta') = \beta'$, we have $\lambda\beta' = \beta'$. Hence $\lambda = 1$. That is,

$$\begin{aligned} g(z) &= z \\ \varphi_\alpha \circ f \circ \varphi_{-\alpha}(z) &= z \\ \implies f(z) &= \varphi_{-\alpha} \circ \varphi_\alpha(z) \\ &= z. \end{aligned}$$

Hence f is the identity.

Before going to the next problem, let us have a definition which will be using shortly. For $z, w \in \mathbb{D}$, define

$$\rho(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

PROBLEM 3 (Schwarz-Pick Theorem). Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic map. Then

$$\rho\big(f(z), f(w)\big) \le \rho(z, w) \qquad \forall\, z, w \in \mathbb{D}.$$

Furthermore,

$$\frac{\big|f'(z)\big|}{1-\big|f(z)\big|^2} \le \frac{1}{1-|z|^2} \qquad \forall\, z \in \mathbb{D}.$$

SOLUTION 3. For $z, w \in \mathbb{D}$, define a map g given by,

$$g(\zeta) = \varphi_{f(w)} \circ f \circ \varphi_{-w}(\zeta).$$

Then, notice that $g : \mathbb{D} \longrightarrow \mathbb{D}$ is holomorphic and $g(0) = 0$. By Schwarz's lemma, we have,

$$\big|g(\zeta)\big| \le |\zeta| \qquad \forall\, \zeta \in \mathbb{D}. \tag{1}$$

Let $z' \in \mathbb{D}$ be such that $z' = \varphi_w(z)$.

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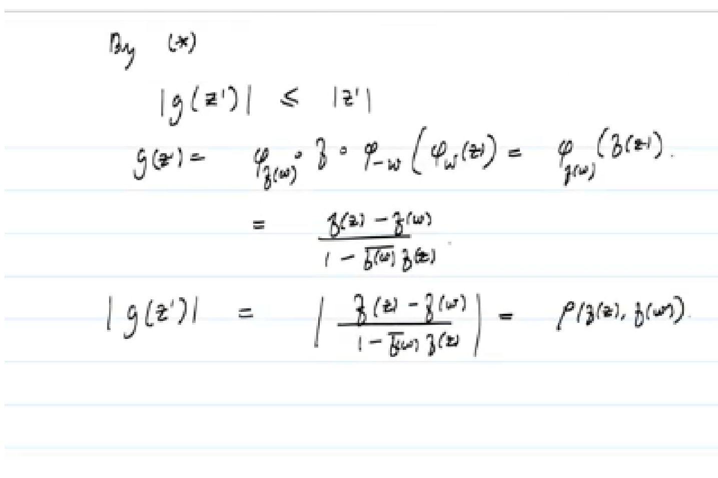

By (1), $\big|g(z')\big| \le |z'|$. Note that,

$$\begin{aligned} g(z') &= \varphi_{f(w)} \circ f \circ \varphi_{-w}\big(\varphi_w(z)\big) \\ &= \varphi_{f(w)}\big(f(z)\big) \\ &= \frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}. \end{aligned}$$

Then,

$$\left|g(z')\right| = \left|\frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}\right| = \rho\left(f(z), f(w)\right).$$

Also,

$$|z'| = \left|\varphi_w(z)\right| = \left|\frac{z-w}{1-\overline{w}z}\right| = \rho(z,w).$$

Hence,

$$\rho\left(f(z), f(w)\right) \leq \rho(z,w).$$

It is an exercise for the reader to check that, if f is an automorphism of $\mathbb{D}$, then the above inequality is an equality.

Thus, for all $z, w \in \mathbb{D}$, we have

$$\left|\frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}\right| \leq \left|\frac{z-w}{1-\overline{w}z}\right|.$$

For $z \neq w$,

$$\left|\frac{f(z)-f(w)}{z-w}\right|\left|\frac{1}{1-\overline{f(w)}f(z)}\right| \leq \left|\frac{1}{1-\overline{w}z}\right|.$$

By taking the limit as $w \longrightarrow z$, we have

$$\left|f'(z)\right| \cdot \frac{1}{1-\left|f(z)\right|^2} \leq \frac{1}{1-|z|^2}.$$

PROBLEM 4. Let f be a non-constant entire function such that $|f(z)| = 1$ for all $|z| = 1$. Describe f.

SOLUTION 4. By the maximum modulus principle, we have $|f(z)| \leq 1$ for each $z \in \mathbb{D}$.

Claim: f has at least one zero in $\mathbb{D}$.

If f does not have a zero in $\mathbb{D}$, then $\frac{1}{f}$ is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

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$\left|1/f(z)\right| = 1$ on $|z| = 1$.

we have max. mod. principle,

$\left|\frac{1}{f(z)}\right| \leq 1$ on $\mathbb{D}$.

$|f(z)| \geqslant 1$. on $\mathbb{D}$.

Note that, $\left|\frac{1}{f(z)}\right| = 1$ for $|z|$, and hence by maximum modulus principle,

$$\left|\frac{1}{f(z)}\right| \leq 1 \qquad \forall\, z \in \mathbb{D}.$$

Hence $|f(z)| \geq 1$ on $\mathbb{D}$. Since we already have $|f(z)| \leq 1$ on $\mathbb{D} \implies |f(z)| = 1$ on $\mathbb{D}$. By open mapping theorem, f is constant, which is a contradiction. Therefore f has at least one zero in $\mathbb{D}$.

Claim: f has finitely many zeroes in $\mathbb{D}$.

If f has infinitely many zeroes in $\overline{\mathbb{D}}$, then the zeroes of f has a limit point by compactness of $\overline{\mathbb{D}}$. Then, by the identity theorem, we have $f \equiv 0$, which is a contradiction as f is non-constant. Hence f has only finitely many zeroes in $\mathbb{D}$.

Let $\alpha_1, \ldots, \alpha_n$ be zeroes of f of order $d_1, \ldots, d_n$ in $\mathbb{D}$. Then, we have

$$f(z) = (z - \alpha_1)^{d_1} \cdots (z - \alpha_n)^{d_n} h(z),$$

where h is a non-vanishing holomorphic function defined on $\mathbb{D}$.

Consider the automorphism of unit disc $\varphi_{\alpha_j}(z) = \dfrac{z - \alpha_j}{1 - \bar{\alpha}_j z}$. Since φ_{α_j} being an automorphism it has unique zero at α_j of order 1. Hence, we have

$$\varphi_{\alpha_j} = (z - \alpha_j)\psi_j(z),$$

where ψ_j is non-vanishing on $\mathbb{D}$. Then $\left(\varphi_{\alpha_j}\right)^{d_j} = \left(z - \alpha_j\right)^{d_j} \left(\psi_j(z)\right)^{d_j}$.

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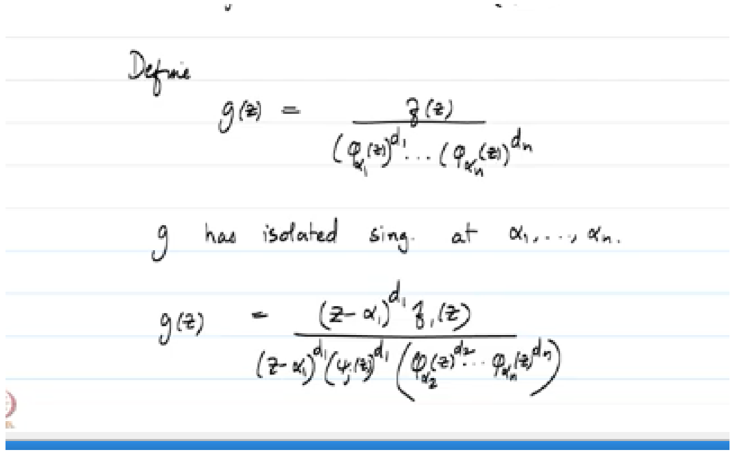

Define

$$g(z) = \frac{f(z)}{\left(\varphi_{\alpha_1}\right)^{d_1} \cdots \left(\varphi_{\alpha_n}\right)^{d_n}}.$$

Then, note that g has isolated singularity at $\alpha_1, \ldots, \alpha_n$.

Since α_1 is a zero of f order d_1, we have

$$g(z) = \frac{(z-\alpha_1)^{d_1} f_1(z)}{(z-\alpha_1)^{d_1} \left(\psi_1(z)\right)^{d_1} \left(\left(\varphi_{\alpha_2}(z)\right)^{d_2} \cdots \left(\varphi_{\alpha_n}(z)\right)^{d_n}\right)}.$$

Now, it is an exercise for the reader to check that g has a removable singularity at α_1. Similarly, prove that g has a removable singularity at α_j for $1 \le j \le n$.

For $|z| = 1$, we have

$$\left|g(z)\right| = \left|\frac{f(z)}{\left(\varphi_{\alpha_1}\right)^{d_1} \cdots \left(\varphi_{\alpha_n}\right)^{d_n}}\right| = \frac{|f(z)|}{\left|\varphi_{\alpha_1}\right|^{d_1} \cdots \left|\varphi_{\alpha_n}\right|^{d_n}} = 1.$$

Since g does not vanish in $\mathbb{D}$ (why?), by a similar argument above, we have $g(z)$ is a constant function. Hence $g(z) = \lambda$, where $|\lambda| = 1$. Thus,

$$f(z) = \lambda \left(\varphi_{\alpha_1}(z)\right)^{d_1} \cdots \left(\varphi_{\alpha_n}(z)\right)^{d_n}.$$