Complex Analysis

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Lecture No – 43

Phragmen-Lindelöf Method

Now we will take a closer look at the maximum modulus principle. The Phragmen-Lindelöf method is, in some sense, a generalization of the maximum modulus principle to the unbounded domains in the complex plane.

Let Ω be a bounded open connected set in $\mathbb{C}$ and $f : \overline{\Omega} \longrightarrow \mathbb{C}$ be continuous on $\overline{\Omega}$ and holomorphic on Ω. If f is non-constant, then by the open mapping theorem, there does not exists $z_1 \in \Omega$ such that f attains a local maximum at z_1. Since f is continuous on $\overline{\Omega}$, there exists $z_0 \in \overline{\Omega}$ such that

$$|f(z_0)| = \sup_{z \in \overline{\Omega}} |f(z)|.$$

Since $z_0 \not\in \Omega$, we have $z_0 \in \partial\Omega$.

Hence $\sup_{z \in \Omega} |f(z)| = \sup_{z \in \partial\Omega} |f(z)|$. If the supremum is attained at an interior, then f is constant.

But, when we ask a similar question on an unbounded domain, we do not have such an answer. In fact, this exact statement, when considered in unbounded domains, is going to be false. Let us discuss an example to illustrate that this particular statement will not go through for unbounded domains.

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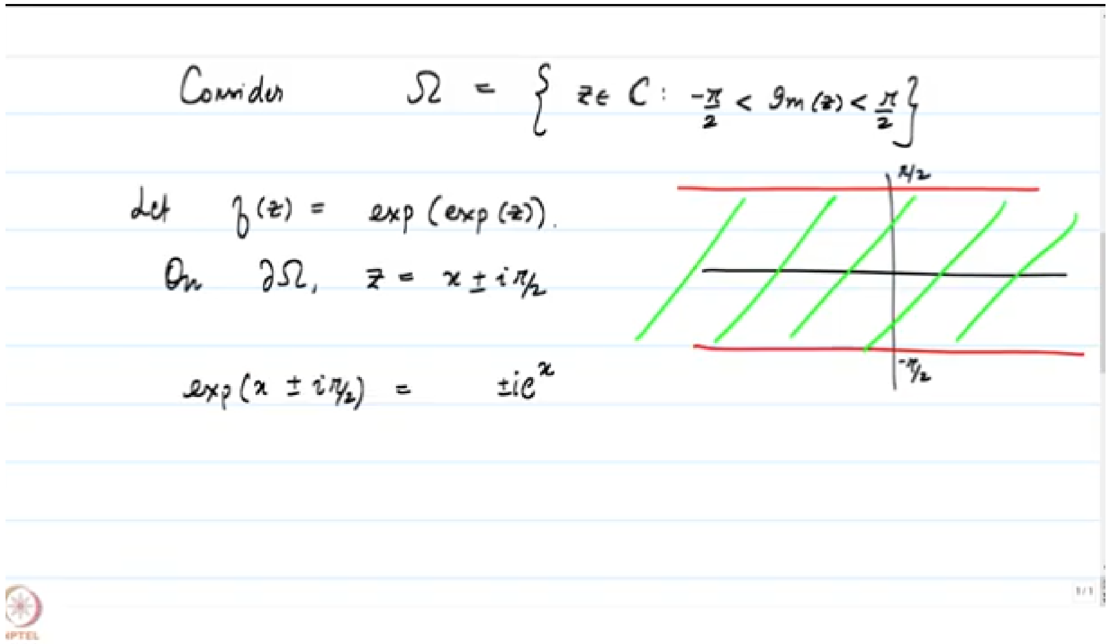

Consider $\Omega = \left\{ z \in \mathbb{C} : \frac{-\pi}{2} < \mathfrak{Im}(z) < \frac{\pi}{2} \right\}$. Then notice that Ω is an unbounded set, which is an infinite strip that contains all points in between the horizontal ray that passes through $i\frac{-\pi}{2}$ and the horizontal ray that passes through $i\frac{\pi}{2}$. In this region, consider the function f given by $f(z) = \exp(\exp(z))$. Then, on the boundary of this domain, an element $z \in \partial\Omega$ will be of the form $z = x \pm i\frac{\pi}{2}$. Observe that

$$\exp\left(x \pm i\frac{\pi}{2}\right) = \pm i e^{x}$$

and hence

$$\left| f\left(x \pm i\frac{\pi}{2}\right) \right| = \left| \exp\left(\pm i e^{x}\right) \right| = 1.$$

That is, on the boundary, $|f|$ is 1. But,

$$\lim_{x \to \infty} |f(x)| = \infty.$$

Clearly, the maximum modulus principle fails in this particular setup. So, the fact that Ω is bounded was playing a crucial role.

We dropped the condition that Ω is bounded and we ended up with this kind of a situation. So, we will ask the next question, what is the next best thing that can be done?

That is where the Phragmen-Lindelöf method comes into our picture. But before going to that, let us describe a couple of application of the method.

When we looked at the Liouville's theorem, it told us that bounded entire functions are necessarily constant. But if we impose the following condition: suppose f is entire and $|f(z)| \leq M + |z|^{1/2}$, then we claim that all entire functions with this growth condition will also be constant. The proof is actually very similar to how we proved Liouville's theorem.

On $D(0,R)$, by Cauchy estimates, we have

$$\left|f^{(n)}(0)\right| \leq \frac{n!\left(M+R^{1/2}\right)}{R^n}.$$

As $R \to \infty$, we have $f^{(n)}(0) = 0$ for each $n \geq 1$. Since f is an entire function, f has power series expansion around 0. That is,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

around 0. But, then $a_n = \frac{f^n(0)}{n!} = 0$ for each $n \geq 1$. Hence $f(z) = a_0$.

That is, even with a weaker growth condition like this, we will be able to conclude the same result as Liouville's theorem.

THEOREM 1. *Let* $\Omega = \{z \in \mathbb{C} : a < \mathfrak{Re}(z) < b\}$. *Let* $f : \overline{\Omega} \longrightarrow \mathbb{C}$ *be such that* f *is continuous on* $\overline{\Omega}$ *and holomorphic on* Ω. *Suppose* $\left|f(z)\right| < B$, *for some large* $B > 0$, *and let*

$$M(x) = \sup\left\{\left|f(x+iy)\right| : -\infty < y < \infty\right\}.$$

Then,

$$M(x)^{b-a} \leq M(a)^{b-x} M(b)^{x-a}.$$

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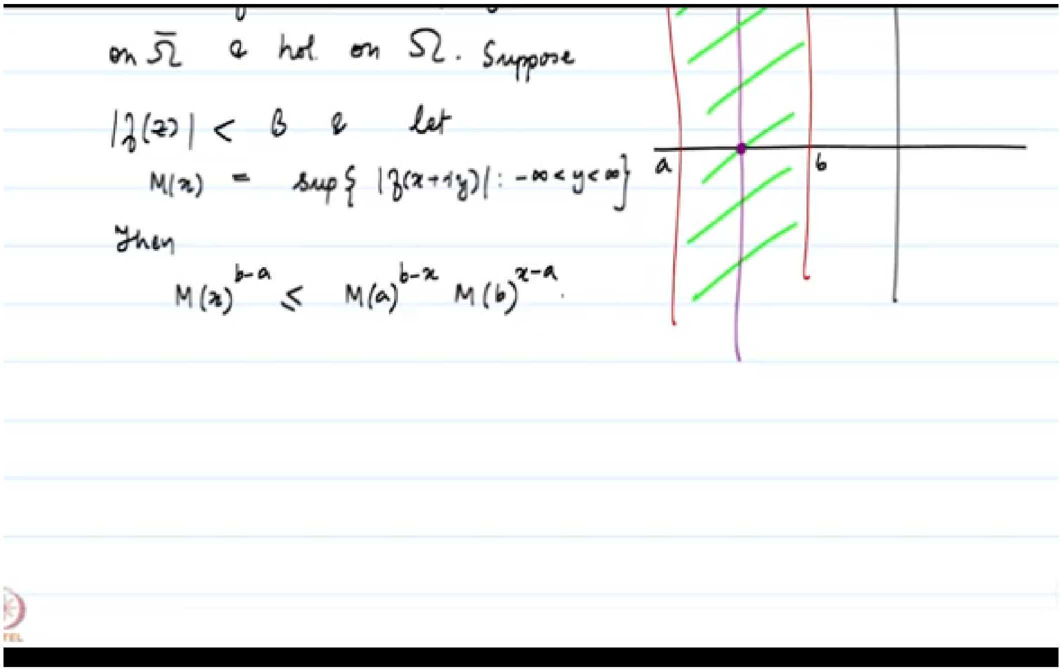

The theorem tells us that, for $z \in \Omega$, $z = x + iy$, then

$$|f(z)| \leq M(x) \leq \max\{M(a), M(b)\} = \sup_{z \in \partial\Omega} |f(z)|.$$

PROOF. Let us assume that $M(a) = M(b) = 1$. For $\epsilon > 0$, define an auxiliary function h_ϵ to be,

$$h_\epsilon(z) = \frac{1}{1 + \epsilon(z-a)}.$$

Note that,

$$|1 + \epsilon(z-a)| > \mathfrak{Re}(1 + \epsilon(z-a)) = 1 + \epsilon(x-a) \geq 1.$$

Hence,

$$|h_\epsilon(z)| \leq 1 \qquad \forall z \in \overline{\Omega}.$$

By our assumption that $M(a) = 1 = M(b)$, for each $z \in \partial\Omega$, we have $|f(z)| \leq 1$. Hence,

$$\left| f(z) h_\epsilon(z) \right| \leq 1 \qquad \forall z \in \partial\Omega.$$

Consider the set $E = \{z = x + iy \in \Omega : |y| \geq \frac{B}{\epsilon}\}$, then $\frac{B}{\epsilon|y|} \leq 1$. If $z = x + iy \in E$, then observe that $\mathfrak{Im}(1 + \epsilon(z-a)) = \epsilon y$

$$\Longrightarrow |1 + \epsilon(z-a)| \geq \epsilon|y|$$

$$\Longrightarrow |h_\epsilon(z)| \leq \frac{1}{\epsilon|y|}$$

$$\Longrightarrow \left|f(z)h_\epsilon(z)\right| \leq \frac{B}{\epsilon|y|} \leq 1.$$

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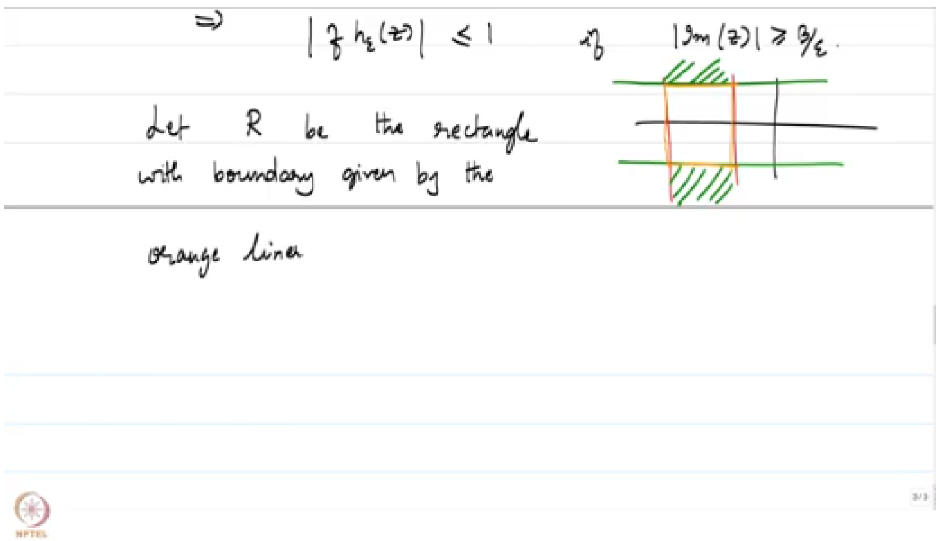

Let R be the rectangle on $\mathbb{C}$ bounded by the lines $x = a, x = b, y = \frac{B}{\epsilon}$ and $y = \frac{-B}{\epsilon}$.

Observe that $\left|f(z)h_\epsilon(z)\right| \leq 1$ for $z \in \partial R$ and hence by the maximum modulus principle, we have $\left|f(z)h_\epsilon(z)\right| \leq 1$ for $z \in R$. Since $\left|f(z)h_\epsilon(z)\right| \leq 1$ for $z \in E$, and by the above observation,

$$\left|f(z)h_\epsilon(z)\right| \leq 1 \qquad \text{for } z \in \overline{\Omega}.$$

Now, as $\epsilon \longrightarrow 0$, we have $f(z)h_\epsilon(z) \longrightarrow f(z)$. Hence $\left|f(z)\right| \leq 1$.

That is, with our assumption that $M(a) = M(b) = 1$, we have

$$M(x) \leq 1 \implies M(x)^{b-a} \leq 1 = 1^{b-x} \cdot 1^{x-a} = M(a)^{b-x} M(b)^{x-a}.$$

Now, let us try to prove the more general case.

Define

$$g(z) = \exp\left(\frac{b-z}{b-a}\ln M(a)\right)\exp\left(\frac{z-a}{b-a}\ln M(b)\right).$$

Now, let us see what happens when we look at g on the boundary of Ω.

$$\begin{aligned}|g(a+iy)| &= \left|\exp\left(\frac{b-(a+iy)}{b-a}\ln M(a)\right)\right|\left|\exp\left(\frac{(a+iy)-a}{b-a}\ln M(b)\right)\right| \\ &= \left|\exp\left(\left(1-i\frac{y}{b-a}\right)\ln M(a)\right)\right|\left|\exp\left(i\frac{y}{b-a}\ln M(b)\right)\right| \\ &= \left|\exp\left(\ln M(a)+i\left(\frac{-y}{b-a}\ln M(a)\right)\right)\right|\cdot 1 \\ &= \exp(\ln(M(a))) = M(a).\end{aligned}$$

Similarly,

$$|g(b+iy) = M(b).$$

Now, it is an exercise for the reader to check that $\frac{1}{g}$ is bounded on $\overline{\Omega}$.

Consider $\frac{f}{g}$ and then we have $\left|\frac{f(z)}{g(z)}\right| \leq 1$ for each $z \in \overline{\Omega}$.

From here one could conclude the result in the general case. (Exercise.) □

So, what did we do in this entire theorem? We somehow introduced an auxiliary function h_ϵ and using h_ϵ, we reduced the problem partly to a problem of the maximum modulus principle being applied on some bounded function a bounded domain. And that is precisely what is referred to as the Phragmen-Lindelöf method.

Let us look at one more example of the Phragmen-Lindelöf method on another unbounded domain which now is going to be a horizontal strip rather than a vertical strip.

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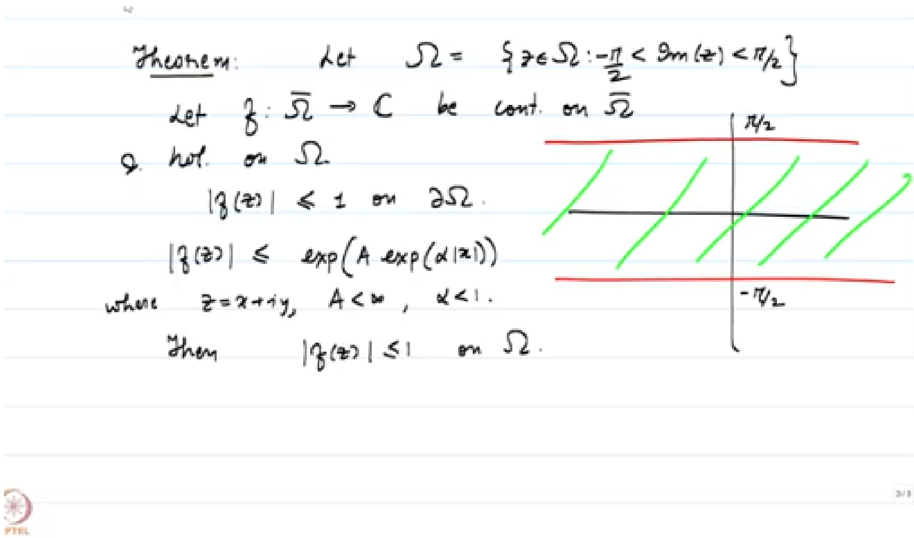

THEOREM 2. *Let* $\Omega = \left\{z \in \mathbb{C} : -\frac{\pi}{2} < \mathfrak{Im}(z) < \frac{\pi}{2}\right\}$. *Let* $f : \overline{\Omega} \longrightarrow \mathbb{C}$ *be continuous on* $\overline{\Omega}$ *and holomorphic on* Ω. *Suppose* $|f(z)| \leq 1$ *for* $z \in \partial\Omega$, $|f(z)| \leq \exp(A \cdot \exp(\alpha|x|))$, *where* $z = x + iy$, $A < \infty$ *and* $\alpha < 1$, *then* $|f(z)| \leq 1$ *for each* $z \in \Omega$.

PROOF. Let $\beta > 0$ be such that $\alpha < \beta < 1$. For $\epsilon > 0$, define an auxiliary function $h_\epsilon(z)$ to be

$$h_\epsilon(z) := \exp\left(-\epsilon\left(e^{\beta z} + e^{-\beta z}\right)\right).$$

Now, notice that for $z = x + iy \in \Omega$,

$$\mathfrak{Re}\left(-\epsilon\left(e^{\beta z} + e^{-\beta z}\right)\right) = -\epsilon\left(e^{\beta x} + e^{-\beta x}\right)\cos\beta y.$$

Since $\frac{-\pi}{2} < y < \frac{\pi}{2}$, and $\beta < 1$, we have

$$\cos(\beta y) > \cos\left(\beta\frac{\pi}{2}\right) = \delta.$$

Hence, on $\overline{\Omega}$, we have

$$\begin{aligned}
|h_\epsilon(z)| &= \left|\exp\left(-\epsilon\left(e^{\beta x}+e^{-\beta x}\right)\cos\beta y\right)\right|\left|\exp\left(i\epsilon\left(e^{-\beta x}-e^{\beta x}\right)\sin\beta y\right)\right| \\
&= \exp\left(-\epsilon\left(e^{\beta x}+e^{-\beta x}\right)\cos\beta y\right) \\
&= \exp\left(-\epsilon\left(e^{\beta x}+e^{-\beta x}\right)\cos\beta\right) \\
&< \exp\left(-\epsilon\delta\left(e^{\beta x}+e^{-\beta x}\right)\right) \\
&< 1
\end{aligned}$$

Consider the function fh_ϵ, then $|f(z)h_\epsilon(z)| \le 1$ for $z \in \partial\Omega$. For $z \in \Omega$,

$$\begin{aligned}
|f(z)h_\epsilon(z)| &\le \exp(A\cdot\exp(\alpha|x|))\exp\left(-\epsilon\delta\left(e^{\beta x}+e^{-\beta x}\right)\right) \\
&= \exp\left(A\cdot\exp(\alpha|x|)-\epsilon\delta\left(e^{\beta x}+e^{-\beta x}\right)\right).
\end{aligned}$$

Check that $\left(A\cdot\exp(\alpha|x|)-\epsilon\delta\left(e^{\beta x}+e^{-\beta x}\right)\right) \longrightarrow -\infty$ as $x \longrightarrow \infty$ or $x \longrightarrow -\infty$.

Hence $|f(z)h_\epsilon(z)| < 1$ for $|x| \ge x_0$ for some $x_0 > 0$.

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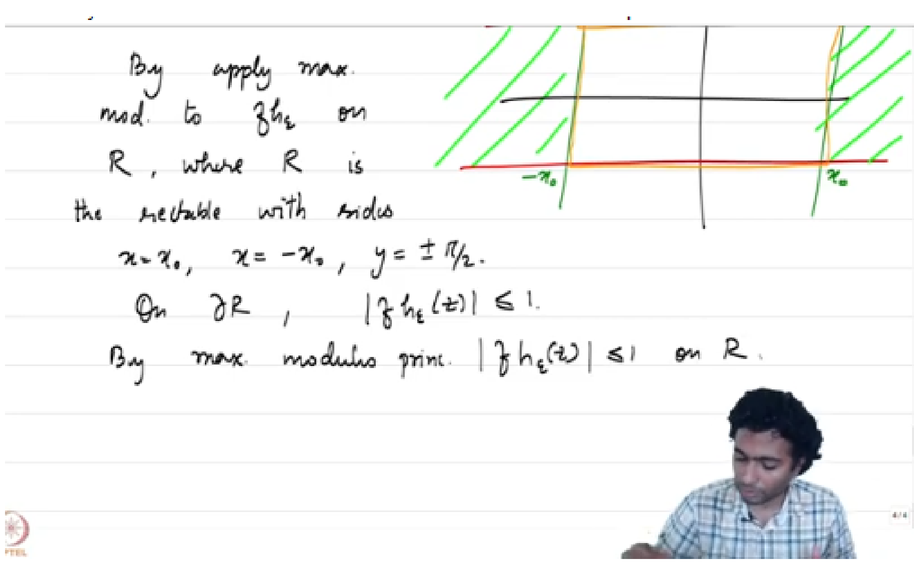

Let R the rectangle with sides $x = \pm x_0$ and $y = \pm\frac{\pi}{2}$. Then on ∂R, we have $|f(z)h_\epsilon(z)| \le 1$. By the maximum modulus principle, $|f(z)h_\epsilon(z)| \le 1$ on R.

Therefore, $|f(z)h_\epsilon(z)| \leq 1$ for every $z \in \overline{\Omega}$. Since $\epsilon > 0$ was arbitrary, $h_\epsilon(z) \longrightarrow 1$ as $\epsilon \longrightarrow 0$ for each $z \in \overline{\Omega}$. Hence $|f(z)| \leq 1$ on $\overline{\Omega}$. $\square$