Complex Analysis

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Lecture No – 42

Automorphisms of the Unit Disk

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Recall that the maximum modulus principle states that if $f : \Omega \to \mathbb{C}$ be a hol. fn on an open set $\Omega \subseteq \mathbb{C}$ & K be a compact subset of Ω, then

$$|f(z)| \leq \sup_{z \in \partial K} |f(z)| \quad \forall \, z \in K.$$

We have already seen the maximum modulus principle for holomorphic functions. It states that if $f : \Omega \longrightarrow \mathbb{C}$ is a holomorphic function on an open set $\Omega \subseteq \mathbb{C}$ and $K \subseteq \Omega$ is a compact set, then

$$\sup_{z \in K} |f(z)| \leq \sup_{z \in \partial K} |f(z).$$

Here we will be giving a phenomenal application of the maximum modulus principle in proving the very famous Schwarz's lemma and thereafter, we will give a characterization of the automorphisms of the unit disk. Automorphisms are holomorphic functions from the unit disk to itself, which has a holomorphic inverse.

LEMMA 1 (Schwarz's Lemma). *Let* $f : \mathbb{D} \longrightarrow \mathbb{D}$ *be a holomorphic function such that* $f(0) = 0$. *Then* $|f(z)| \leq |z|$ *for every* $z \in \mathbb{D}$ *and* $|f'(0)| \leq 1$.
Furthermore, if $|f(z)| = |z|$ *for some* $z \in \mathbb{D} \setminus \{0\}$ *or if* $|f'(0)| = 1$, *then there exists* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$ *such that* $f(z) = \lambda z$.

PROOF. Since $f(0) = 0$, we have $g(z) = \frac{f(z)}{z}$ has a removable singularity at 0.

Let $0 < r < 1$. By the maximum modulus principle on $\overline{D(0,r)}$ applied to g, we have

$$
\begin{aligned}
|g(z)| &\leq \frac{\sup\limits_{|z|=r} |f(z)|}{r} \qquad \forall\, z \in \overline{D(0,r)} \\
&\leq \frac{1}{r}.
\end{aligned}
$$

Since this is true for $0 < r < 1$, taking limit $r \to 1$, we have $|g(z)| \leq 1$ for each $z \in D(0,1)$. That is,

$$\frac{|f(z)|}{|z|} \leq 1 \qquad \text{for every } z \in D(0,1).$$

Hence,

$$|f(z) \leq |z| \qquad \text{for every } z \in D(0,1).$$

Notice that $f'(0) = g(0)$ and since $|g(0)| \leq 1$ we have $|f'(0)| \leq 1$.

If $|g(z)| = 1$ for some $z \in \mathbb{D}$, then g is a constant function. That is,

$$g(z) = \lambda, \qquad \text{for every } z \in \mathbb{D} \text{ and for some } \lambda \text{ such that } |\lambda| = 1.$$

Hence, $|f(z)| = \lambda z$ for every $z \in \mathbb{D}$. Similarly, if $|g(0)| = 1$, we have $f(z) = \lambda z$ for each $z \in \mathbb{D}$ and some λ such that $|\lambda| = 1$. □

Now, let us consider some special functions on the unit disk $\mathbb{D}$. For $\alpha \in \mathbb{D}$, define $\varphi_\alpha : \mathbb{D} \longrightarrow \mathbb{C}$ given by,

$$\varphi_\alpha(z) := \frac{z - \alpha}{1 - \bar{\alpha} z}.$$

Notice that φ_α is a Möbius transformation with a pole on $\frac{1}{\bar{\alpha}}$. But since $\alpha \in \mathbb{D}$, we have $\frac{1}{\bar{\alpha}} \notin \mathbb{D}$. Hence this function is holomorphic in a neighborhood of the closure of the unit disk for each $\alpha \in \mathbb{D}$.

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$$\varphi_\alpha\left(\varphi_{-\alpha}(z)\right) = \frac{\frac{z+\alpha}{1+\bar{\alpha}z} - \alpha}{1 - \bar{\alpha}\,\frac{z+\alpha}{1+\bar{\alpha}z}}$$

$$= \frac{z+\alpha-\alpha-|\alpha|^2 z}{1+\bar{\alpha}z - \bar{\alpha}z - |\alpha|^2} = \frac{(1-|\alpha|^2)z}{(1-|\alpha|^2)} = z$$

Now,

$$\begin{aligned}
\varphi_\alpha\left(\varphi_{-\alpha}(z)\right) &= \frac{\frac{z+\alpha}{1+\bar{\alpha}z} - \alpha}{1 - \bar{\alpha}\frac{z+\alpha}{1+\bar{\alpha}z}} \\
&= \frac{z+\alpha-\alpha-|\alpha|^2 z}{1+\bar{\alpha}z-\bar{\alpha}z-|\alpha|^2} \\
&= \frac{\left(1-|\alpha|^2\right)z}{\left(1-|\alpha|^2\right)} \\
&= z.
\end{aligned}$$

Hence, $\varphi_{-\alpha}$ and φ_α are holomorphic functions which are inverses of each other. Observe that,

$$\begin{aligned}
\left|\varphi_\alpha\left(e^{i\theta}\right)\right| &= \left|\frac{e^{i\theta}-\alpha}{1-\bar{\alpha}e^{i\theta}}\right| \\
&= \frac{\left|e^{i\theta}-\alpha\right|}{\left|e^{i\theta}\right|\left|e^{-i\theta}-\bar{\alpha}\right|} \\
&= \frac{\left|e^{i\theta}-\alpha\right|}{\left|e^{i\theta}-\alpha\right|} \\
&= 1.
\end{aligned}$$

That is, $\left|\varphi_\alpha(z)\right| = 1$ for each $z \in \partial\mathbb{D}$. By maximum modulus principle, we have $\left|\varphi_\alpha(z)\right| \le 1$ for each $z \in \mathbb{D}$. That is image of φ_α is contained in $\mathbb{D}$, also, since it is not a constant function $\varphi_\alpha(\mathbb{D})$ will be an open subset of $\mathbb{D}$. By a very similar argument, we have $\left|\varphi_{-\alpha}(z)\right| \le 1$ for each $z \in \mathbb{D}$. Combining these two facts, we have $\varphi_\alpha(\mathbb{D}) = \mathbb{D}$.

That is, $\varphi_\alpha : \mathbb{D} \longrightarrow \mathbb{D}$ is a holomorphic map of $\mathbb{D}$ to itself which has a holomorphic inverse.

DEFINITION 1 (Automorphism). We say that a function $f : \Omega \longrightarrow \Omega$ is an automorphism if f is holomorphic and has a holomorphic inverse.

By what we have done above, φ_α is an automorphism of $\mathbb{D}$ for each $\alpha \in \mathbb{D}$. Observe that φ_α has some nice properties like,

$$\varphi_\alpha(\alpha) = 0, \qquad \varphi_\alpha(0) = -\alpha, \qquad \varphi_{-\alpha}(0) = \alpha.$$

Also,

$$\varphi'_\alpha(0) = 1 - |\alpha|^2 \qquad \varphi'_\alpha(\alpha) = \frac{1}{1-|\alpha|^2}.$$

Now, let us try to answer the following question:

Question: Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic function such that $f(\alpha) = \beta$, where $\alpha, \beta \in \mathbb{D}$. What is the maximum value of $|f'(\alpha)|$?

That is, we are asking for $\sup\left\{|f'(\alpha)| : f : \mathbb{D} \longrightarrow \mathbb{D} \text{ holomorphic and } f(\alpha) = \beta\right\}$.

Define $g(z) = \varphi_\beta \circ f \circ \varphi_{-\alpha}(z)$. Then g is a holomorphic function, $g : \mathbb{D} \longrightarrow \mathbb{D}$ and $g(0) = 0$. Then by Schwarz's lemma, $\left|g'(0)\right| \leq 1$.

Check that, by chain rule,

$$g'(0) = \varphi'_\beta(\beta) f'(\alpha) \varphi'_{-\alpha}(0).$$

Then,

$$\left|g'(0)\right| = \frac{1}{1-\left|\beta\right|^2}\left|f'(\alpha)\right|\left(1-|\alpha|^2\right) \leq 1.$$

$$\implies |f'(\alpha)| \leq \frac{1-\left|\beta\right|^2}{1-|\alpha|^2}.$$

This is the upper bound of what $|f'(\alpha)|$ can have in terms of α and β. Let us now check whether we can attain this value of some function.

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If $|g'(0)| = 1 \Rightarrow g(z) = \lambda z$

$\Rightarrow \varphi_\beta \circ f \circ \varphi_{-\alpha}(z) = \lambda z.$

$\Rightarrow f(z) = \varphi_{-\beta}(\lambda \varphi_\alpha(z))$

Consider $f : \mathbb{D} \to \mathbb{D}$
given by $f(z) = \varphi_{-\beta}(\lambda\, \varphi_\alpha(z)).$
& $|f'(\alpha)| = 1 - |\beta|^2$

If $|g'(0)| = 1$, then by Schwarz's lemma $g(z) = \lambda z$ for some λ with $|\lambda| = 1$. That is,

$$g(z) = \lambda z$$
$$\implies \varphi_\beta \circ f \circ \varphi_{-\alpha}(z) = \lambda z$$
$$\implies f(z) = \varphi_{-\beta}\big(\lambda\varphi_\alpha(z)\big).$$

Hence if we consider the function $f : \mathbb{D} \longrightarrow \mathbb{D}$ given by $f(z) = \varphi_{-\beta}\big(\lambda\varphi_\alpha(z)\big)$, then

$$\left|f'(\alpha)\right| = \frac{1 - \left|\beta\right|^2}{1 - |\alpha|^2}.$$

THEOREM 2. *Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be an automorphism. Then there exist $\alpha \in \mathbb{D}$ and $\lambda \in \partial\mathbb{D}$ such that $f(z) = \lambda\varphi_\alpha(z)$.*

PROOF. Since f is an automorphism, there exists $\alpha \in \mathbb{D}$ such that $f(\alpha) = 0$. Let $g\mathbb{D} \longrightarrow \mathbb{D}$ be the holomorphic function such that

$$g\big(f(z)\big) = z \qquad \text{for } z \in \mathbb{D}.$$

By chain rule, we have

$$g'(0)f'(\alpha) = 1. \tag{1}$$

We know from the previous discussion that,

$$\left|g'(0)\right| \le 1-|\alpha|^2$$

and

$$|f'(\alpha) \le \frac{1}{1-|\alpha|^2}.$$

Hence

$$\left|g'(0)f'(\alpha)\right| \le 1.$$

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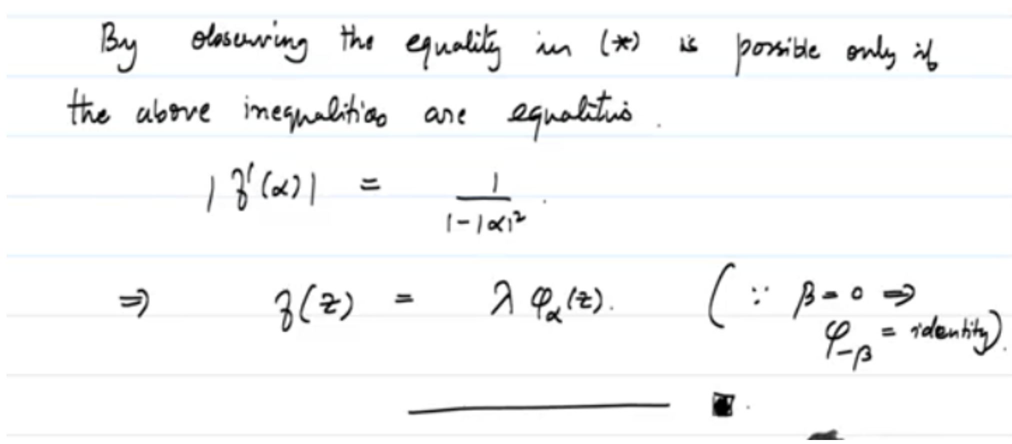

By observing the equality in (1), it is possible only if the above inequalities are equalities. Then,

$$|f'(\alpha)| = \frac{1}{1-|\alpha|^2}$$

and it is possible only if

$$f(z) = \lambda\varphi_\alpha(z).$$

□