Complex Analysis

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Lecture No – 41

Branch of the Logarithm

When we proved the argument principle, we defined the notion of the log-derivative of a holomorphic function f as the meromorphic function $\frac{f'}{f}$. We even checked that the log-derivative satisfies some of the properties, which the complex logarithm is expected to satisfy. We did all that without really defining what is meant by the complex logarithm in the complex setting. We are familiar with the same notion in the real setting. However, the notion has to be made precise in the complex setting.

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This definition cannot be generalized to define the complex logarithm since $\exp : \mathbb{C} \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Notice that

$e^z = e^w$ iff $z = w + 2\pi i k$ for $k \in \mathbb{Z}$.

exp is not injective.

Recall that the logarithm $\ln : (0, \infty) \longrightarrow \mathbb{R}$ to be the function which inverts the exponential function on $\mathbb{R}$. This definition cannot be generalized to define the complex logarithm. Notice that the exponential function $\exp : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$ satisfy the property that

$$e^z = e^w \iff z = w + 2\pi i k \text{ for } k \in \mathbb{Z}.$$

Thus exp is not injective.

For $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, let us denote $\log(z)$ to be the set

$$\log(z) = \{w \in \mathbb{C} : \exp(w) = z\}.$$

If $w = \ln|z| + i\theta$ for $\theta \in \arg(z) = \{\theta \in \mathbb{R} : z = |z|e^{i\theta}\}$, then $e^w = |z|e^{i\theta} = z$.

That is an alternate way of looking at it, but this is not something which we will be satisfied with. We would like to really get hold of an 'honest' function f, which inverts the exponential function. Let us try to define one such function and in order to do that let us revisit from the real analysis setting; what we did, when we encountered real valued functions?

Let us consider a function on $\mathbb{R}$ which is not injective. One such function is $f : \mathbb{R} \longrightarrow [0, \infty)$ given by $f(x) = x^2$. Given $x > 0$, there exist two real numbers $\sqrt{x}, -\sqrt{x}$ such that $(\sqrt{x})^2 = (-\sqrt{x})^2 = x$. By picking one of the square roots, we worked freely with the square root.

By a branch of the square root, we mean a function $g : [0, \infty) \longrightarrow \mathbb{R}$ such that $(g(x))^2 = x$.

Can we have a g which is continuous?

Define $g(x) = \sqrt{x}$, where $\sqrt{x} > 0$, then g is continuous branch of the square root. We could also define other branches as well. For example, define $g(x) = -\sqrt{x}$ where $\sqrt{x} > 0$, then also g is continuous branch of the square root.

Observe that, we could also define a branch of the square root which is not continuous. For example, define,

$$g(x) := \begin{cases} \sqrt{x} & 0 \leq x \leq 10 \\ -\sqrt{x} & x > 10 \end{cases}$$

then g will be a branch of the square root, but g is not a continuous function as g has a discontinuity at $x = 10$.

DEFINITION 1 (Branch of the logarithm). Let Ω be an open connected subset of $\mathbb{C}^*$. By a branch of the logarithm on Ω, we mean a function $f : \Omega \longrightarrow \mathbb{C}$ such that

$$\exp\big(f(z)\big) = z \qquad \text{for each } z \in \Omega$$

.

Let $\Omega = \mathbb{C} \setminus \{0\} = \mathbb{C}^*$. Suppose f is a branch of logarithm on $\mathbb{C}^*$. That is $f : \mathbb{C}^* \longrightarrow \mathbb{C}$ be a function such that $\exp\big(f(z)\big) = z$.

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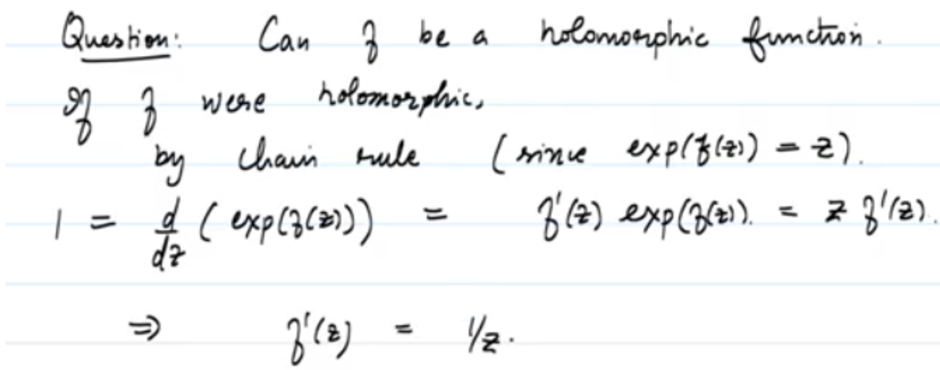

One can ask this question: Can f be a holomorphic function?

If f were holomorphic, then by chain rule applied to the equation $\exp\big(f(z)\big) = z$, we have

$$1 = \frac{d}{dz}\big(\exp\big(f(z)\big)\big) = f'(z)\exp\big(f(z)\big) = zf'(z)$$

$$\implies f'(z) = \frac{1}{z}.$$

Let $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$ in $\mathbb{C}^*$. Since f is the anti-derivative of $\frac{1}{z}$ in $\mathbb{C}^*$, by the fundamental theorem of calculus, we have

$$\int_\gamma f'(z)\,dz = 0.$$

But,

$$\int_\gamma f'(z)\,dz = \int_\gamma \frac{dz}{z} = 2\pi i \neq 0.$$

Hence if we have a holomorphic branch of the logarithm on $\mathbb{C}^*$, then we have a contradiction.

Let Ω be an open connected subset of $\mathbb{C}^*$. Further let Ω be simply connected. Consider the function $\frac{1}{z}$ which is holomorphic on Ω. Let γ be any simply closed curve. By Cauchy's theorem,

$$\int_\gamma \frac{dz}{z} = 0.$$

By the fundamental theorem of calculus, there exists a function $f : \Omega \longrightarrow \mathbb{C}$ such that $f'(z) = \frac{1}{z}$.

Let f be an anti-derivative such that given z_0 and w_0 with $e^{z_0} = w_0$, we have $f(w_0) = z_0$. Since f is holomorphic on Ω, by chain rule

$$\frac{d}{dz}\left(\exp\left(f(z)\right)\right) = f'(z)\exp\left(f(z)\right) = \frac{\exp\left(f(z)\right)}{z}.$$

Now, by the quotient rule

$$\frac{d}{dz}\left(\frac{\exp\left(f(z)\right)}{z}\right) = \frac{\exp\left(f(z)\right) - \exp\left(f(z)\right)}{z^2} = 0 \qquad \forall\, z \in \Omega.$$

$$\implies \frac{\exp\left(f(z)\right)}{z} = C, \text{ a constant.}$$

For $z = w_0$, we have

$$\exp\left(f(w_0)\right) = \exp(z_0) = w_0.$$

Hence

$$\frac{\exp\left(f(w_0)\right)}{w_0} = C = 1.$$

$$\implies \exp\left(f(z)\right) = z.$$

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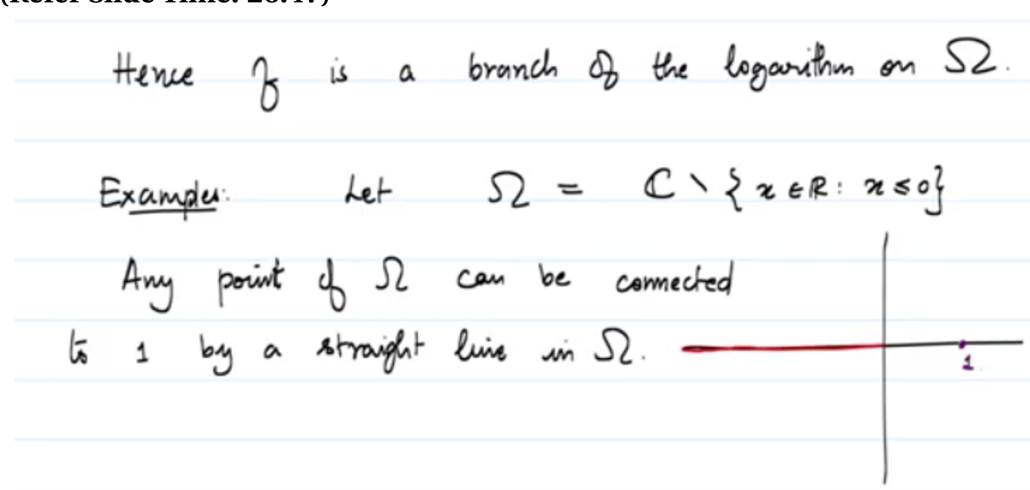

Hence f is a branch of the logarithm on Ω.

EXAMPLE 1. Let $\Omega = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$. Any point of Ω can be connected to 1 be a straight line in Ω. If γ is a closed curve in Ω, then

$$H(s,t) = (1-s)\gamma(t) + s$$

is a homotopy of closed curves in Ω from γ to the constant curve γ_1. Hence Ω is simply connected.

Therefore, there exists a holomorphic function $f : \mathbb{C} \setminus \{x \leq 0\} \longrightarrow \mathbb{C}$ such that $\exp\big(f(z)\big) = z$.

Suppose f and g are two branches of the logarithm on Ω which are continuous functions. We know that $\exp\big(f(z)\big) = \exp\big(g(z)\big)$ for each $z \in \Omega$. That is, for each $z \in \Omega$, we have $\exp\big(f(z) - g(z)\big) = 1$. Hence $f(z) = g(z) = 2\pi i k$ for $k \in \mathbb{Z}$. Since f and g are continuous functions on Ω, we have $\big(f - g\big)$ is continuous on Ω and hence $\big(f - g\big)(\Omega)$ is connected. Note that the only connected subsets of $\{2\pi i k : k \in \mathbb{Z}\}$ are singletons (Why?). Since the image of $f - g$ will be connected and are $2\pi i k$ for $k \in \mathbb{Z}$, we must have a fixed $k_0 \in \mathbb{Z}$ such that

$$\big(f - g\big)(\Omega) = 2\pi i k_0.$$

That is,

$$f(z) = g(z) + 2\pi i k_0 \qquad \text{for some fixed } k_0 \in \mathbb{Z}.$$

Recall that $\operatorname{Arg}(z) = \theta$, where $\theta \in (-\pi, \pi]$ and $z = |z|e^{i\theta}$.

DEFINITION 2 (Standard branch of Logarithm). Let $\Omega = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$. Define the standard branch of the logarithm to be

$$\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z).$$

EXERCISE 2. $\operatorname{Log}(z)$ is continuous on $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$ and also $\exp\big(\operatorname{Log}(z)\big) = z$.

Hence $\mathrm{Log}(z)$ is a continuous branch of the logarithm on Ω.

Since there exists a holomorphic branch of the logarithm f on Ω, by the observation made above, we have

$$\mathrm{Log}(z) = f(z) + 2\pi i k_0 \qquad \text{for some fixed } k_0 \in \mathbb{Z}.$$

Hence $\mathrm{Log}(z)$ is a holomorphic function.

Notice that $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$ is just one example of the simply connected domain in $\mathbb{C}^*$ for which we have a holomorphic branch of the logarithm. We can get hold of another examples of simple domains in $\mathbb{C}^*$ that admits a continuous branch of the logarithm.

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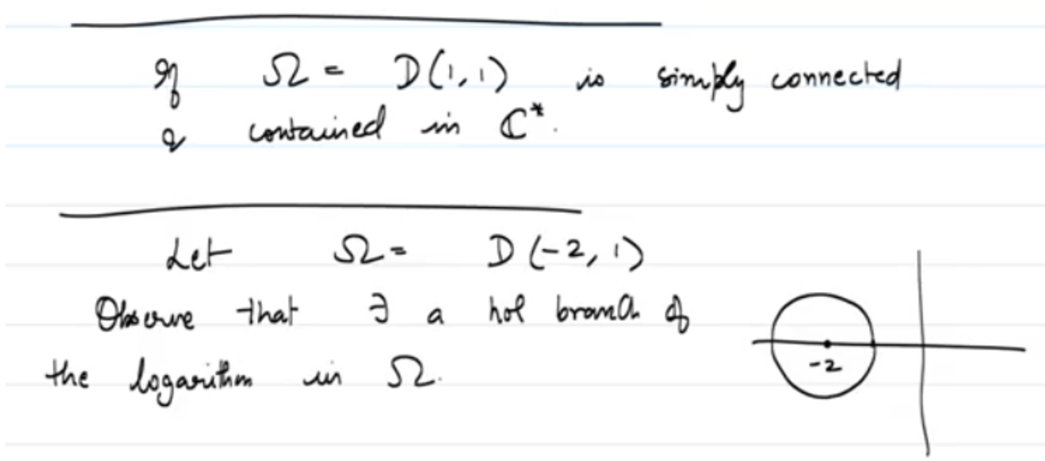

If $\Omega = D(1,1)$, then Ω is simply connected and contained in $\mathbb{C}^*$. Hence in this domain also, we have a continuous branch of the logarithm and moreover, since $D(1,1)$ is contained in $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$ as well, the restriction of the holomorphic branch of the logarithm $\mathrm{Log}(z)$ to $D(1,1)$ will again turn out to a holomorphic function on $D(1,1)$.

If $\Omega = D(-2,1)$, then Ω is a simply connected domain in $\mathbb{C}^*$ and hence there exists a holomorphic branch of the logarithm in Ω. However, the $\mathrm{Log}(z)$ (which can be defined on $\mathbb{C}^*$) is not even continuous in Ω as $\mathrm{Log}(z)$ is discontinuous on $\{x \in \mathbb{R} : x \leq 0\} \cap \Omega$.

Thus the point here to notice is that, when we talk about the holomorphic branch of the complex logarithm, it matters which domain we are considering.

Recall that, given $w \in \mathbb{C}^*$, there exists n roots to the equation $z^n = w$.

We would like to obtain a holomorphic branch of the n^{th} root function. That is, we want to address the question: does there exists a holomorphic function $g : \Omega \longrightarrow \mathbb{C}$ such that $\left(g(z)\right)^n = z$ for each $z \in \Omega$, where $\Omega \subseteq \mathbb{C}^*$?

Let Ω be a simply connected domain. and let f be a holomorphic branch of the logarithm on Ω. Define

$$g(z) = \exp\left(\frac{f(z)}{n}\right).$$

Being a composition of holomorphic functions on Ω, g is holomorphic on Ω. Also,

$$\left(g(z)\right)^n = \left(\exp\left(\frac{f(z)}{n}\right)\right)^n = \exp\left(f(z)\right) = z$$

and this is precisely what we are trying to get hold off.

Hence, on simply connected domains, we have a holomorphic branch of n^{th} root function.