Complex Analysis

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Lecture No – 40

Problem Session

PROBLEM 1. Compute the integral

$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$$

where $a > 1$.

SOLUTION 1. Let $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. Now, for $z \in \gamma$, we have $|z| = 1$. If $z = e^{\theta}$, then $\cos\theta = \dfrac{z+\bar{z}}{2} = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$.

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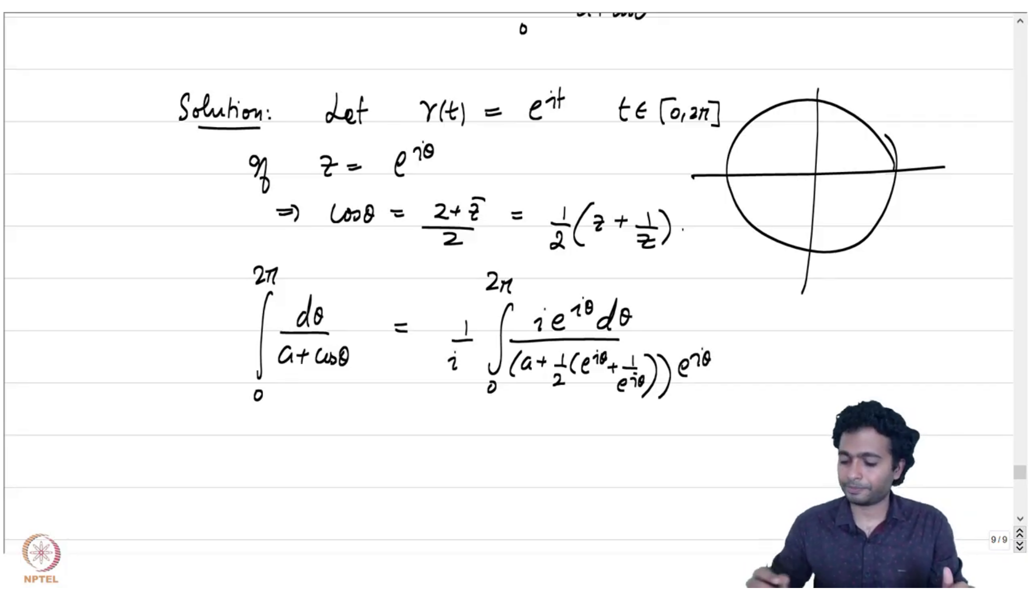

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} &= \frac{1}{i}\int_0^{2\pi} \frac{ie^{i\theta}d\theta}{\left(a+\frac{1}{2}\left(e^{i\theta}+\frac{1}{e^{i\theta}}\right)\right)e^{i\theta}} \\
&= \frac{1}{i}\int_0^{2\pi} \frac{\gamma'(t)dt}{\left(a+\frac{1}{2}\left(\gamma(t)+\frac{1}{\gamma(t)}\right)\right)\gamma(t)} \\
&= \frac{1}{i}\int_\gamma \frac{dz}{\left(a+\frac{1}{2}\left(z+\frac{1}{z}\right)\right)z} \\
&= \frac{2}{i}\int_\gamma \frac{dz}{z^2+2az+1}.
\end{aligned}$$

We know that $z^2+2az+1$ has roots at $-a\pm\sqrt{a^2-1}$. Note that, since $a>1, -a-\sqrt{a^2-1}<-1$ and hence $-a-\sqrt{a^2-1}\notin\overline{\mathbb{D}}$. Also it is an easy verification that $-1<-a+\sqrt{a^2+1}<0$.

By residue theorem, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{dz}{z^2+2az+1} &= W_\gamma(-a-\sqrt{a^2-1})\mathrm{Res}\left(f,-a-\sqrt{a^2-1}\right)+ \\
&\quad W_\gamma(-a+\sqrt{a^2-1})\mathrm{Res}\left(f,-a+\sqrt{a^2-1}\right) \\
&= \mathrm{Res}\left(f,-a+\sqrt{a^2-1}\right).
\end{aligned}$$

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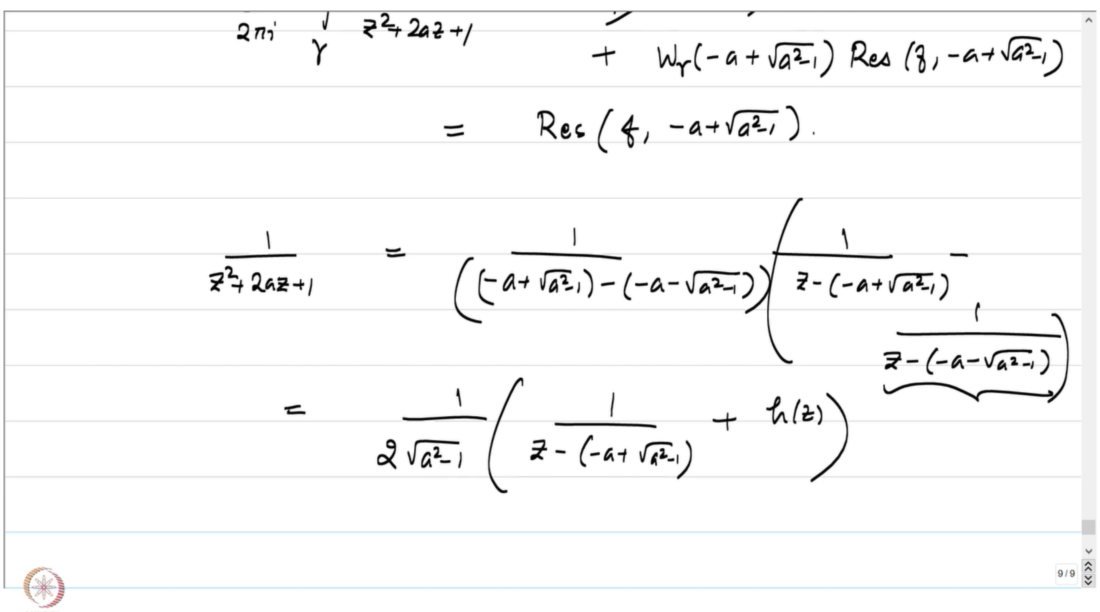

Now, on $\mathbb{D}$

$$\frac{1}{z^2+2az+1} = \left(\frac{1}{\left(-a-\sqrt{a^2-1}\right)-\left(-a+\sqrt{a^2-1}\right)}\right)\left(\frac{1}{z-\left(-a+\sqrt{a^2-1}\right)}-\frac{1}{z-\left(-a-\sqrt{a^2-1}\right)}\right)$$
$$= \frac{1}{2\sqrt{a^2-1}}\left(\frac{1}{z-\left(-a+\sqrt{a^2-1}\right)}+h(z)\right)$$

where h is a function holomorphic on $\mathbb{D}$. Hence, $\operatorname{Res}\left(f,-a+\sqrt{a^2-1}\right)=\frac{1}{2\sqrt{a^2-1}}$.

Therefore,

$$\int_0^{2\pi}\frac{d\theta}{a+\cos\theta}=\frac{2}{i}\int_\gamma\frac{dz}{z^2+2az+1}=4\pi\left(\frac{1}{2\pi i}\int_\gamma\frac{dz}{z^2+2az+1}\right)=\frac{2\pi}{2\sqrt{a^2-1}}.$$

PROBLEM 2. Compute the integral

$$\int_0^\infty\frac{1}{\left(1+x^2\right)^2}dx.$$

SOLUTION 2. Since $\frac{1}{\left(1+x^2\right)^2}$ is an even function,

$$\int_0^\infty\frac{1}{\left(1+x^2\right)^2}dx=\frac{1}{2}\int_{-\infty}^\infty\frac{1}{\left(1+x^2\right)^2}dx=\frac{1}{2}\lim_{R\to\infty}\int_{-R}^R\frac{1}{\left(1+x^2\right)^2}dx.$$

Consider the function $f(z)=\frac{1}{\left(1+z^2\right)^2}=\frac{1}{(z+i)^2(z-i)^2}$.

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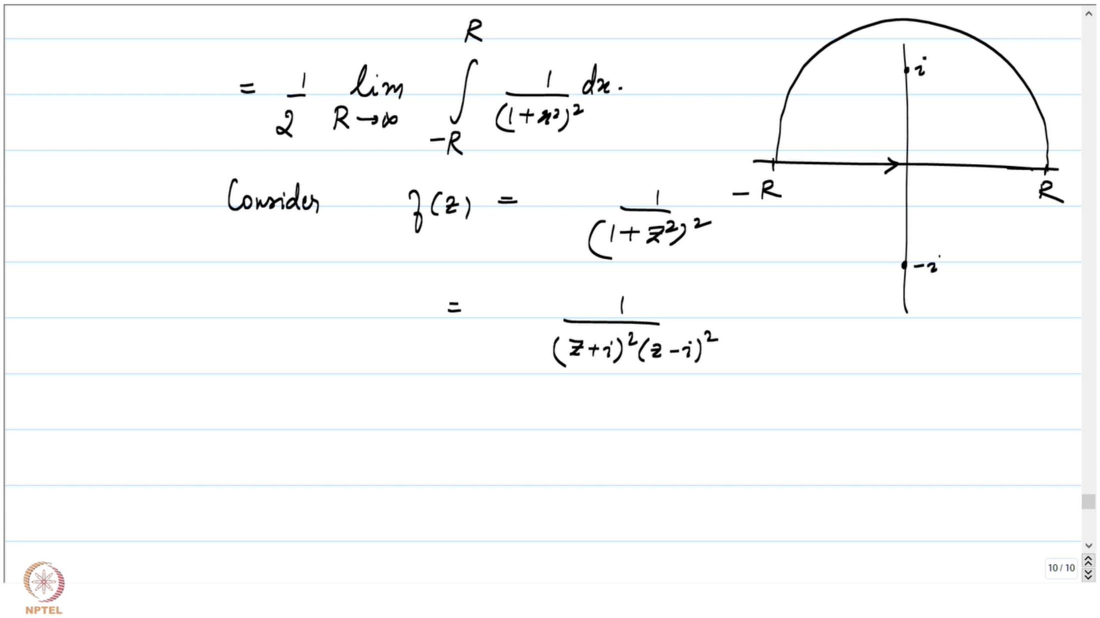

Consider C to be the curve $\gamma_{-R\to R}+\gamma$, where $\gamma(t) = Re^{it}$ for $t \in [0,2\pi]$ and $R \in \mathbb{R}_+$.

Then,

$$\frac{1}{2\pi i}\int_C \frac{1}{\left(1+z^2\right)^2}dz = \text{Res}\left(\frac{1}{\left(1+z^2\right)^2}, i\right).$$

Let us now try to find out what will be $\text{Res}\left(\frac{1}{\left(1+z^2\right)^2}, i\right)$.

We have,

$$\frac{1}{\left(1+z^2\right)^2} = \frac{1}{(z-i)^2}\left(\frac{1}{(z+i)^2}\right).$$

Suppose

$$\frac{1}{(z+i)^2} = \sum_{n=0}^{\infty} b_n (z-i)^n.$$

Then

$$\frac{1}{\left(1+z^2\right)^2} = \frac{b_0}{(z-i)^2} + \frac{b_1}{(z-i)} + h(z)$$

where h is a holomorphic function on i. Therefore,

$$\text{Res}\left(\frac{1}{\left(1+z^2\right)^2}, i\right) = b_1.$$

Since b_1 is the coefficient of $(z-i)$ in the power series expansion of $\frac{1}{(z+i)^2}$ around i,

$$b_1 = \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)\bigg|_{z=i} = \frac{1}{4i}.$$

Thus,

$$\frac{1}{2\pi i}\int_C \frac{1}{\left(1+z^2\right)^2}\,dz = \frac{1}{4i}.$$

Now, notice that

$$\int_C \frac{1}{\left(1+z^2\right)^2}\,dz = \int_{\gamma_{-R\to R}} \frac{1}{\left(1+z^2\right)^2}\,dz + \int_\gamma \frac{1}{\left(1+z^2\right)^2}\,dz$$

and also

$$\int_{\gamma_{-R\to R}} \frac{1}{\left(1+z^2\right)^2}\,dz = \int_{-R}^{R} \frac{1}{\left(1+x^2\right)^2}\,dx.$$

Now, let us concentrate on the term $I = \frac{1}{2\pi i}\int_\gamma \frac{1}{(1+z^2)^2}\,dz$.

By triangle inequality, we have $\left|1+z^2\right| > R^2-1$ and hence $\left|\frac{1}{\left(1+z^2\right)^2}\right| < \frac{1}{R^4-2R^2+1}$.

Therefore,

$$|I| = \left|\frac{1}{2\pi i}\int_\gamma \frac{1}{\left(1+z^2\right)^2}\,dz\right| < \frac{2\pi R}{2\pi}\cdot\frac{1}{R^4-2R^2+1} = \frac{1}{R^3-2R+\frac{1}{R}}$$

and we have $|I| \longrightarrow 0$ as $R \longrightarrow \infty$.

Hence taking the limit as $R \longrightarrow \infty$,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dx}{\left(1+x^2\right)^2} = \frac{1}{4i}.$$

Hence

$$\int_0^\infty \frac{1}{\left(1+x^2\right)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\left(1+x^2\right)^2}\,dx = \frac{\pi}{4}.$$

PROBLEM 3. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{\left(1+x^2\right)^2}\,dx.$$

SOLUTION 3. Here the problem is similar to that of Problem 2. Hence the setup for solving this problem is also similar to that of Problem 2 that we consider the curve $C = \gamma_{-R \to R} + \gamma$, for $\gamma(t) = Re^{it}$ for $t \in [0, 2\pi]$.

If $f(z) = \dfrac{e^{iz}}{\left(1+z^2\right)^2} = \dfrac{1}{(z-i)^2}\left(\dfrac{e^{iz}}{(z+i)^2}\right)$, then $\text{Res}\big(f, i\big)$ will be the coefficient of $(z-i)$ in the power series expansion of $\dfrac{e^{iz}}{(z+i)^2}$ around $z = i$,

$$\text{Res}\left(f, i\right) = \frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right)\Bigg|_{z=i} = \frac{-i}{2e}.$$

Hence,

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{e^{ix}}{\left(1+x^2\right)^2}\,dx = 2\pi i\left(\frac{-i}{2e}\right) = \frac{\pi}{e}.$$

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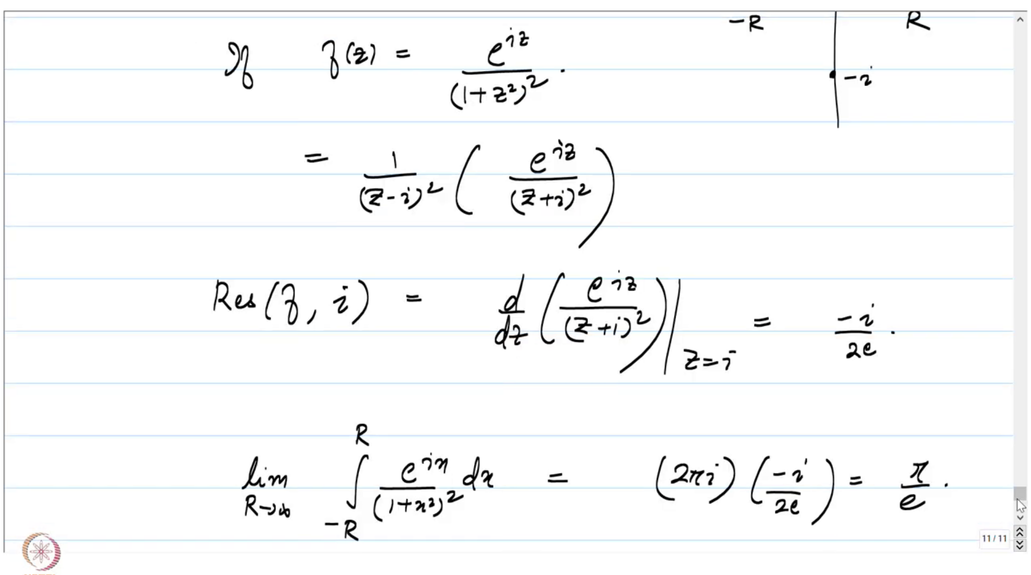

PROBLEM 4. Let Ω be an open set containing $\overline{\mathbb{D}}$ and let f be non-constant holomorphic function on Ω which satisfies $|f(z)| = 1$ whenever $|z| = 1$. Then $\mathbb{D}$ is contained in $f(\Omega)$.

SOLUTION 4. **Claim:** There exists $z_0 \in \mathbb{D}$ such that $f(z_0) = 0$.

If not, then $g(z) = \frac{1}{f(z)}$ is holomorphic in a neighborhood of $\overline{\mathbb{D}}$. Since $|f(z)| = 1$ for $|z| = 1$, by the maximum modulus principle, we have

$$|f(z)| \leq 1 \qquad \forall\, z \in \mathbb{D}. \tag{1}$$

For $|z| = 1$, note that $|g(z)| = 1$. By a similar argument as above, we have $|g(z)| \leq 1$. Hence

$$|f(z)| \geq 1 \qquad \forall\, z \in \mathbb{D}. \tag{2}$$

By (1) and (2), we have $|f(z)| = 1$ for $z \in \mathbb{D}$. By open mapping theorem, $f(\mathbb{D})$ is open and thus $|f(z)| = 1$ is a contradiction. Hence there exists $z_0 \in \mathbb{D}$ such that $f(z_0) = 0$.

Let $w \in \mathbb{D}$. Define a constant function g on Ω given by $g(z) = -w$. Then $|g(z)| < |f(z)|$ on the circle $|z| = 1$. By Rouche's theorem, f and $f + g$ have equal number of zeroes in $\mathbb{D}$ counted with multiplicity.

We know that f vanishes on $\mathbb{D}$. Hence there exists $z_1 \in \mathbb{D}$ such that $(f + g)(z_1) = 0$. That is, $f(z_1) = w$. Since $w \in \mathbb{D}$ was arbitrary, we have $\mathbb{D} \subseteq f(\Omega)$.